Lecture 6 : *Single Queues*

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Classifications of queues

- **Models for ________ state space Markov chains**
- **X / X / X / X / X**

Distribution of ________________ time
(M, D, E_r, H_t, G)

Distribution of ________ time
(M, D, E_r, H_t, G)

No. of ______

Size of ______

Size of ___________

Classifications of queues (cont'd)

- **M: _________ (Memoryless)**
- **D: ____________**
- **E_r: r-stage __________**
- **H_k: k-stage ____________**
- **G: _________**
- **(ex) M/M/1/∞/∞ (M/M/1)**

Queuing discipline

- Based on the order of _______ from the queue
  - FCFS/ LCFS/ RR (Round Robin)/ PS (Processor Sharing)/ Random/ Priority/ SJF/ LJF

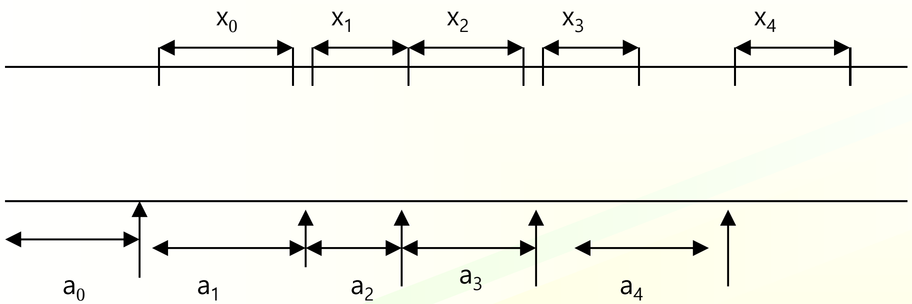
(Similar to RR but time slice is very ______)

(where the new one can stop the one in _________)

- Based on preemption modes for priority or LCFS queue
  - Non-preemptive/ Preemptive-resume/ Preemptive-restart

System utilization

- Utilization(ρ): Fraction of time a system is busy
- Bottleneck: Component with a utilization close to 1



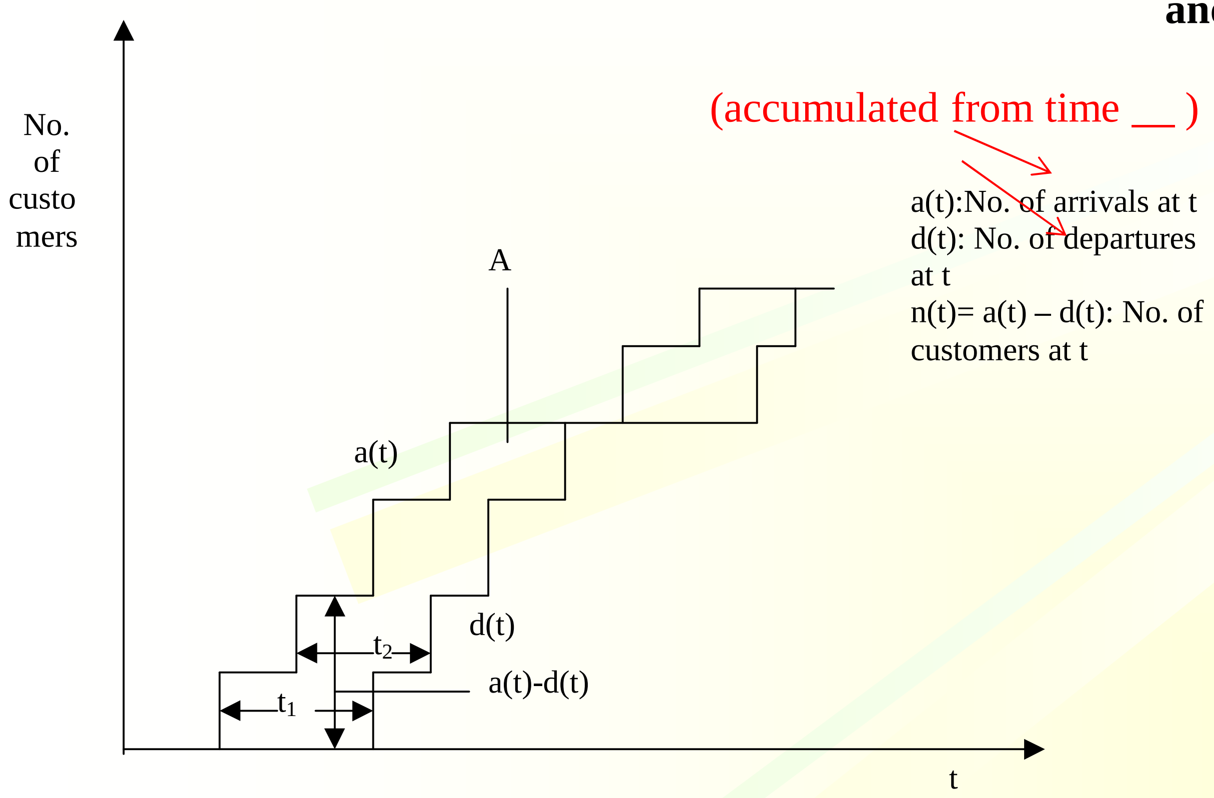
(services)

(job arrivals)

$$\rho = \lim_{N\to\infty} \frac{\sum_{x=1}^{N} x_n}{\sum_{n=1}^{N} a_n} = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} x_n / N}{\sum_{n=1}^{N} a_n / N} = \frac{\bar{x}}{\bar{a}} = \lambda\bar{x}$$

- Multi-server system: $\rho = \dfrac{\lambda\bar{x}}{m}$ (with *m* servers)
- (Ex 6.2) For a queue with 2 servers of service rate of μ respectively and arrival rate of λ, the utilization is ______

Little's theorem



- Assume a(0) = d(0) and a(τ) = d(τ), and *k* customers have arrived during τ

λ = avg. customer arrival rate

$$= \frac{a(\tau)}{\tau} = \frac{k}{\tau}$$

T = avg. delay time per customer

$$= \frac{1}{a(\tau)}\sum_{k=1}^{a(\tau)} t_k$$

N = avg. no. of customers in the system

$$= \frac{1}{\tau}\int_0^{\tau} n(t)dt$$

$$A = \int_0^{\tau}(a(t)-d(t))dt = \sum_{k=1}^{a(\tau)} t_k$$

$$\int_0^{\tau}(n(t))dt = a(\tau)T$$

(holds for any __________ displine)

τN = λτT, <u>N = λT</u>

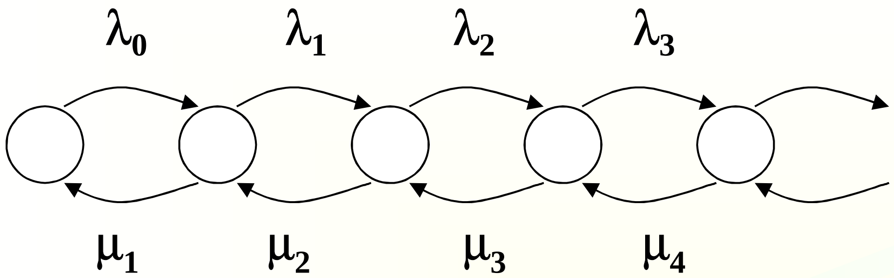
- **Work conserving system** (no work is created or __________ within the system)

Little's theorem (example)

- (Em 6.1) Observes 32 customers per hour arriving on the average and notices that each customer exits after 12 minutes on the average, how many customers stay inside on the average?

- (Ex 6.3) A simulation program has finished the execution of 12.356 jobs while 25.6 jobs arrive on the average per minute. How long each job takes to finish on the average?

Birth-death systems



- __________ time MC

$$\pi Q = 0, \qquad Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & & & \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & 0 & . & . & . \\ 0 & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & & & \\ & & & & & & \\ & & . & & & & \\ & & . & & & & \\ & & . & & & & \end{bmatrix}$$

Birth-death systems (cont'd)

- **In steady state, the state change rate must be __.**
  - $\Rightarrow q_{k,k} = -q_{k,k-1} - q_{k,k+1}$

$$-\pi_0\lambda_0 + \pi_1\mu_1 = 0 \rightarrow \pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$$

$$\pi_0\lambda_0 - (\mu_1 + \lambda_1)\pi_1 + \pi_2\mu_2 = 0$$

$$\pi_0(\lambda_0 - (\mu_1 + \lambda_1)\frac{\lambda_0}{\mu_1}) = -\pi_2\mu_2$$

$$\pi_0(-\frac{\lambda_1\lambda_0}{\mu_1}) = -\pi_2\mu_2 \rightarrow \pi_2 = \frac{\lambda_0\lambda_1}{\mu_2\mu_1}\pi_0$$

$$\vdots$$

$$\pi_k = \frac{\lambda_0\lambda_1\ldots\lambda_{k-1}}{\mu_k\mu_{k-1}\ldots\mu_1}\pi_0 = \left(\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}}\right)\pi_0$$

$$\sum_{k=0}^{\infty}\pi_k = 1 \rightarrow \pi_0\left(1 + \sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}}\right) = 1$$

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}}}, \quad \pi_k = \frac{\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}}}{1 + \sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}}}$$

Birth-death systems (cont'd)

- $$\pi_0 = \frac{1}{1+\sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}}}, \pi_k = \frac{\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}}}{1+\sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}}}$$
- **Ergodicity**
  - **Aperiodic**
  - **Recurrent non-null (check if $\pi_k \neq 0$)**

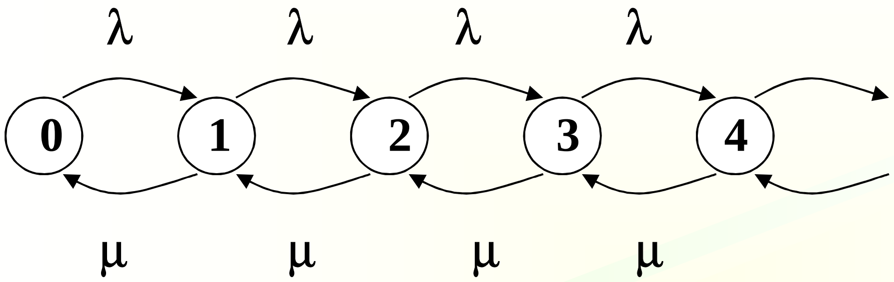
$$\pi_k = A\pi_0 \;;\; \pi_k \neq 0 \text{ if } \pi_0 \neq 0 \text{ and } A \neq 0$$

$$S_0 \equiv \frac{1}{\pi_0} = 1+\sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}} \;;\; S_1 \equiv \sum_{k=1}^{\infty}\frac{1}{\prod_{j=0}^{k-1}\frac{\lambda_j}{\mu_{j+1}}}$$

S_0	S_1	Markov Chain
$< \infty$	$= \infty$	__________
$= \infty$	$= \infty$	Recurrent null
$= \infty$	$< \infty$	__________

  - **For convergence, there must be a k beyond which $\lambda < \mu$**

M/M/1 queue



(constant λ, μ)

- **At steady state**

$$\lambda\pi_0 = \mu\pi_1 \rightarrow \pi_1 = \frac{\lambda}{\mu}\pi_0$$

$$\lambda\pi_1 = \mu\pi_2 \rightarrow \pi_2 = \frac{\lambda}{\mu}\pi_1 = (\frac{\lambda}{\mu})^2\pi_0$$

$$\vdots$$

M/M/1 queue (cont'd)

$$\pi_k = (\frac{\lambda}{\mu})^k \pi_0, \frac{\lambda}{\mu} = \rho$$

$$\pi_k = \rho^k \pi_0$$

$$\sum_{k=0}^{\infty} \pi_k = 1 ; \ \pi_0 \sum_{k=0}^{\infty} \rho^k = 1 ; \ \pi_0 = \frac{1}{\frac{1}{1-\rho}} = 1 - \rho$$

$$N = \sum_{k=0}^{\infty} k\pi_k = (1-\rho) \sum_{k=0}^{\infty} k\rho^k$$

M/M/1 queue (cont'd)

- **Ergodic if $\lambda < \mu$**

$$\frac{d\left(\sum_{k=0}^{\infty}\rho^k\right)}{d\rho} = \frac{d\left(\frac{1}{1-\rho}\right)}{d\rho}$$

$$\sum_{k=0}^{\infty} k\rho^{k-1} = -\frac{1}{(1-\rho)^2}(-1)$$

$$\frac{1}{\rho}\sum_{k=0}^{\infty} k\rho^k = \frac{1}{(1-\rho)^2}$$

$$\sum_{k=0}^{\infty} k\rho^k = \frac{\rho}{(1-\rho)^2}$$

$$N = (1-\rho)\frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

M/M/1 queue (cont'd)

- Another approach for getting N

$$\pi_k = (1-\rho)\rho^k$$

$$\pi^*(z) = \sum_{k=0}^{\infty} \pi_k z^k$$

$$= (1-\rho)\sum_{k=0}^{\infty} (\rho z)^k$$

$$= \frac{1-\rho}{1-\rho z}$$

$$N = \frac{d}{dz}\pi^*(z)\Big|_{z=1} = \frac{\rho(1-\rho)}{(1-\rho z)^2}\Big|_{z=1}$$

$$= \frac{\rho}{1-\rho}$$

- N_Q = Average number of customers in the queue

  $= N - \rho = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho}$

  (* ρ is service utilization which is average number of customers in the service *)

M/M/1 queue (cont'd)

- **Avg no. of customers in service, $E[C]$**

  C: r.v., 1 if a customer in service, 0 otherwise

  $$P[C=1] = \sum_{k=1}^{\infty}(1-\rho)\rho^{k} = 1-\pi_0 = \rho$$

  $$E[C] = 0 \times P[C=0] + 1 \times P[C=1] = \rho$$

- **By Little's theorem**

  $$T = \frac{N}{\lambda} = \frac{\frac{1}{\mu}}{1-\rho};\quad \frac{1}{\mu} = \text{average time in server}$$

  $$T_Q = T - \frac{1}{\mu} = \frac{\frac{1}{\mu}}{1-\rho} - \frac{1}{\mu} = \frac{\frac{\lambda}{\mu^2}}{1-\rho} \quad \text{(average waiting time in queue)}$$

  $= (\rho/(1-\rho))(1/\mu) =$ __ $(1/\mu)$

  (Any incoming job sees __ customers in the system. Thus, it needs to wait $N(1/\mu)$ time for them to ______ the system to get the service.)

M/M/1 queue (cont'd)

- **Prob. density function of waiting time, $w(t)$**

$$W = R + \sum_{i=2}^{k} X_i$$

$$W(t|k) = S_1(t) \otimes S_2(t) \otimes \cdots \otimes S_k(t)$$

$$W^*(s|k) = [S^*(s)]^k = [\frac{\mu}{\mu+s}]^k$$

$$W^*(s) = \sum_{k=0}^{\infty}[\frac{\mu}{\mu+s}]^k \pi_k$$

$$= \sum_{k=0}^{\infty}[\frac{\mu}{\mu+s}]^k (1-\rho)\rho^k$$

$$= (1-\rho)\frac{\mu+s}{(1-\rho)\mu+s}$$

$$= (1-\rho) + (1-\rho)\frac{\rho\mu}{s+(1-\rho)\mu}$$

(Proof)

$$\int_0^\infty w(t)dt = (1-\rho) + \int_0^\infty (1-\rho)\rho\mu e^{-(1-\rho)\mu t}dt$$

$$= (1-\rho) + (1-\rho)\rho\mu \frac{e^{-(1-\rho)\mu t}}{-(1-\rho)\mu}\Big|_0^\infty$$

$$= (1-\rho) + \rho = 1$$

Hence, $w(t)$ is a correct _______ function.

$$w(t) = \begin{cases} 1-\rho & t = 0 \\ (1-\rho)\rho\mu e^{-(1-\rho)\mu t} & t > 0 \end{cases}$$

M/M/1 queue (cont'd)

- $$E[T] = -\frac{d}{ds}F^*(s)\Big|_{s=0} = -(1-\rho)\rho\mu\frac{-1}{(s+(1-\rho)\mu)^2}\Big|_{s=0}$$

$$= \frac{\rho}{(1-\rho)\mu} = \frac{\frac{\lambda}{\mu^2}}{1-\rho} = T_Q \ or$$

$$E[T] = \int_0^\infty tw(t)dt = (1-\rho)\rho\mu\int_0^\infty te^{-(1-\rho)\mu t}dt$$

$$= (1-\rho)\rho\mu\left[\frac{te^{-(1-\rho)\mu t}}{-(1-\rho)\mu}\Big|_0^\infty - \int_0^\infty \frac{e^{-(1-\rho)\mu t}}{-(1-\rho)\mu}dt\right]$$

$$= (1-\rho)\rho\mu\frac{1}{(1-\rho)\mu}\frac{e^{-(1-\rho)\mu t}}{-(1-\rho)\mu}\Big|_0^\infty = \frac{\rho}{(1-\rho)\mu} = T_Q$$

M/M/1 queue (cont'd)

- **(Ex 6.4) M/M/1 queue of arrival of 2 per minute and serve of 4 per minute. How many customers on the average?**

  $\lambda = 2;\ \mu = 4 \quad \rho =$ ____

  $N = \rho/(1-\rho)=1$

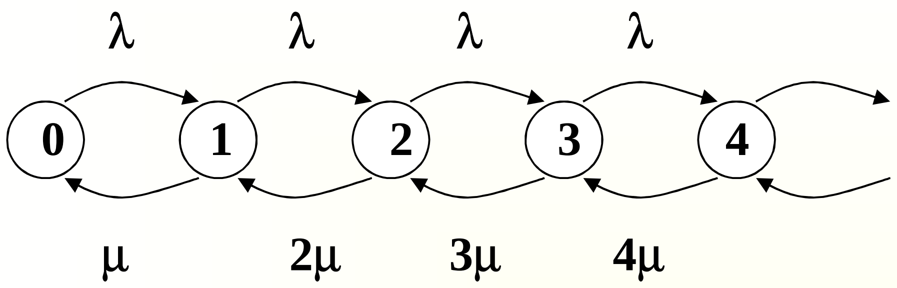
- **(Ex 6.5) M/M/1 queue of 4 people in the queue excluding the one in service. What is the average utilization?**

  $N_Q = \rho^2/(1-\rho) = 4$

  $\rho^2 + 4\rho - 4 = 0$

  $\rho = -2 \pm \sqrt{(4+4)} = -2 \pm 2\sqrt{2} = 0.828$

M/M/∞ queue



Will there be any customer waiting in the queue at any moment?

- **Infinite servers, constant arrival rate λ, constant service rate μ per customer**
- $\mu_k = k\mu$

$$\pi_k = \left(\prod_{j=0}^{k-1} \frac{\lambda}{(j+1)\mu}\right)\pi_0 = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \pi_0$$

$$\mu\pi_1 = \lambda\pi_0, \quad \pi_1 = \frac{\lambda}{\mu}\pi_0$$

$$2\mu\pi_2 = \lambda\pi_1, \quad \pi_2 = \frac{\lambda}{2\mu}\frac{\lambda}{\mu}\pi_1$$

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$$\pi_0 = \frac{1}{1+\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k} = \frac{1}{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k} = \frac{1}{e^{\frac{\lambda}{\mu}}} = e^{-\frac{\lambda}{\mu}}$$

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!}e^{-\frac{\lambda}{\mu}} \quad : \textbf{Poisson density}$$

M/M/∞ queue (cont'd)

- **Ergodic if** $\frac{\lambda}{\mu} < \infty$ **since**

$$S_0 = e^{\frac{\lambda}{\mu}} < \infty; \quad S_1 = \sum_{k=0}^{\infty} \frac{k!}{(\frac{\lambda}{\mu})^k} = \infty$$

- **Calculation of *N* and *T***

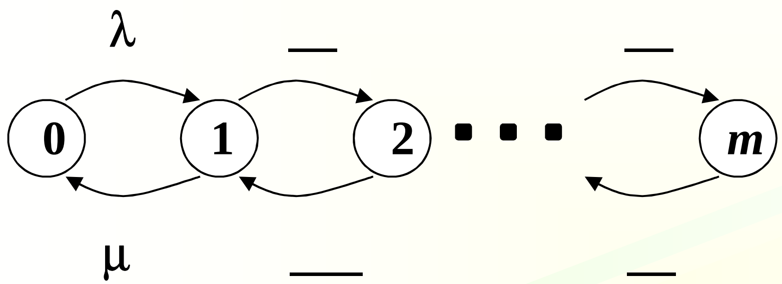
$$\pi^*(z) = e^{-\frac{\lambda}{\mu}} \sum_{k=0}^{\infty} \frac{(\frac{\lambda z}{\mu})^k}{k!} = e^{-\frac{\lambda}{\mu}} e^{\frac{\lambda z}{\mu}} = e^{\frac{\lambda(z-1)}{\mu}}$$

$$N = \frac{d}{dz}\pi^*(z)\Big|_{z=1} = \frac{\lambda}{\mu} e^{\frac{\lambda(z-1)}{\mu}}\Big|_{z=1} = \frac{\lambda}{\mu}$$

$$T = \frac{N}{\lambda} = \frac{1}{\mu}$$

Exercise

- (Ex 6.6) ***m*-server loss queue. Solve for the steady-state probability of *k* customers being in the system.**



$$\mu\pi_1 = \lambda\pi_0,\ \ \pi_1 = \frac{\lambda}{\mu}\pi_0$$

$$2\mu\pi_2 = \lambda\pi_1,\ \ \pi_2 = \frac{\lambda}{2\mu}\pi_1 = \frac{\lambda}{2\mu}\frac{\lambda}{\mu}\pi_0$$

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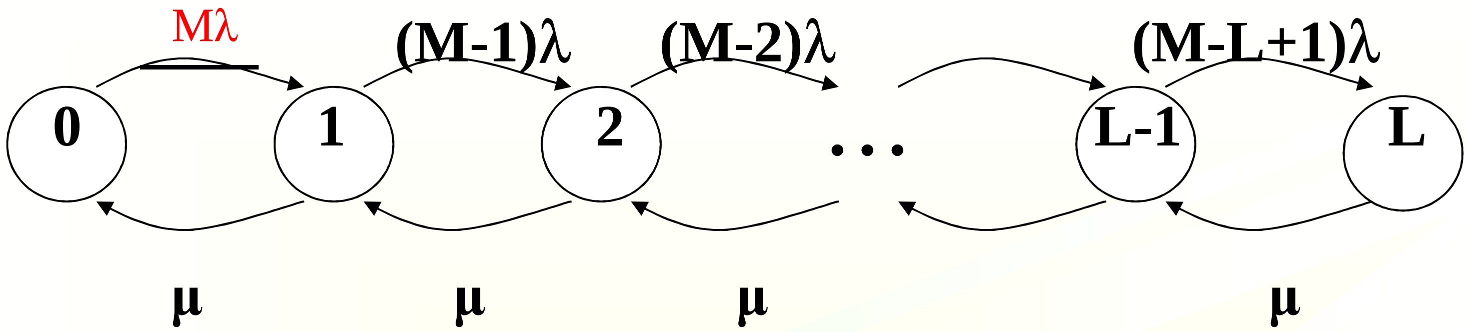
$$\pi_k = (\qquad\qquad)\pi_0,\ \ k \le m$$

$$\pi_0\left(1+\sum_{k=1}^{m}\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k\right)=1 \quad \pi_0\left(\sum_{k=0}^{m}\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k\right)=1$$

$$\pi_0 = \frac{1}{\sum_{k=0}^{m}\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k}$$

M/M/1/L/M queue (M > L)



- $\lambda_k = \begin{cases} (M-k)\lambda & k < L \\ 0 & k \geq L \end{cases}$

  $\mu_k = \mu$

$$\pi_k = \left(\prod_{j=0}^{k-1} \frac{(M-j)\lambda}{\mu}\right)\pi_0 = \frac{M!}{(M-k)!}\left(\frac{\lambda}{\mu}\right)^k \pi_0$$

$$\pi_0 = \frac{1}{1 + \sum_{j=1}^{L} \frac{M!}{(M-j)!}\left(\frac{\lambda}{\mu}\right)^j}$$

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$$\mu\pi_1 = M\lambda\pi_0, \quad \pi_1 = \frac{M\lambda}{\mu}\pi_0$$

$$\mu\pi_2 = (M-1)\lambda\pi_1, \quad \pi_2 = \frac{(M-1)\lambda}{\mu}\pi_1 = \frac{M\lambda}{\mu}\frac{(M-1)\lambda}{\mu}\pi_0$$

•••

$$\pi_k = \frac{\frac{\left(\frac{\lambda}{\mu}\right)^k}{(M-k)!}}{\sum_{j=0}^{L} \frac{\left(\frac{\lambda}{\mu}\right)^j}{(M-j)!}}; \quad N = \sum_{k=1}^{L} k\pi_k$$

Exercise

- (Ex 6.7) Consider (M/M/1/L/∞) queue. Find the probability of there *k* customers being in the system.

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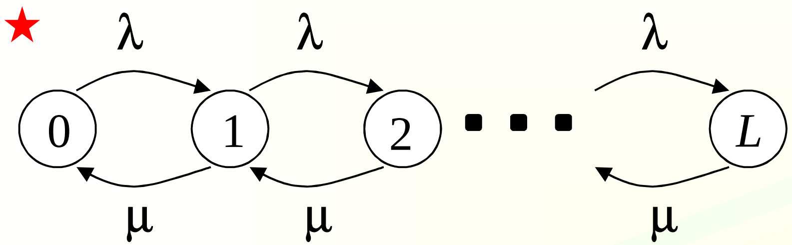

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$$\mu\pi_1 = \lambda\pi_0, \quad \pi_1 = \frac{\lambda}{\mu}\pi_0$$

$$\mu\pi_2 = \lambda\pi_1, \quad \pi_2 = (\frac{\lambda}{\mu})\pi_1 = (\frac{\lambda}{\mu})^2\pi_0$$

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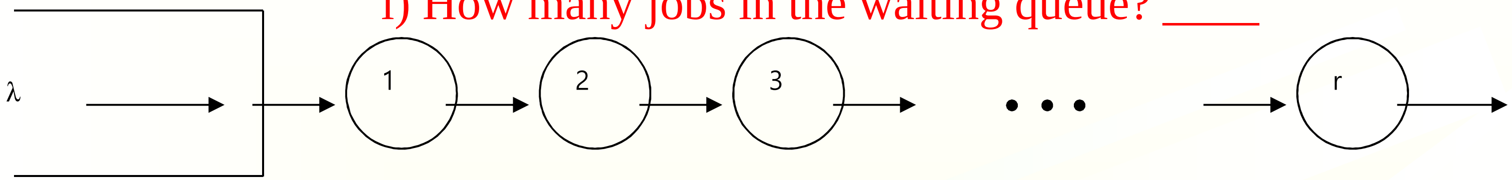
$$\pi_k = (\qquad)\pi_0, \quad k \le L$$

$$(\frac{\lambda}{\mu})^0\pi_0 + (\frac{\lambda}{\mu})^1\pi_0 + (\frac{\lambda}{\mu})^2\pi_0 + \dots + (\frac{\lambda}{\mu})^L\pi_0 = 1 \qquad \pi_0(\sum_{j=0}^{L}(\frac{\lambda}{\mu})^j) = 1$$

Non-Birth-Death Systems

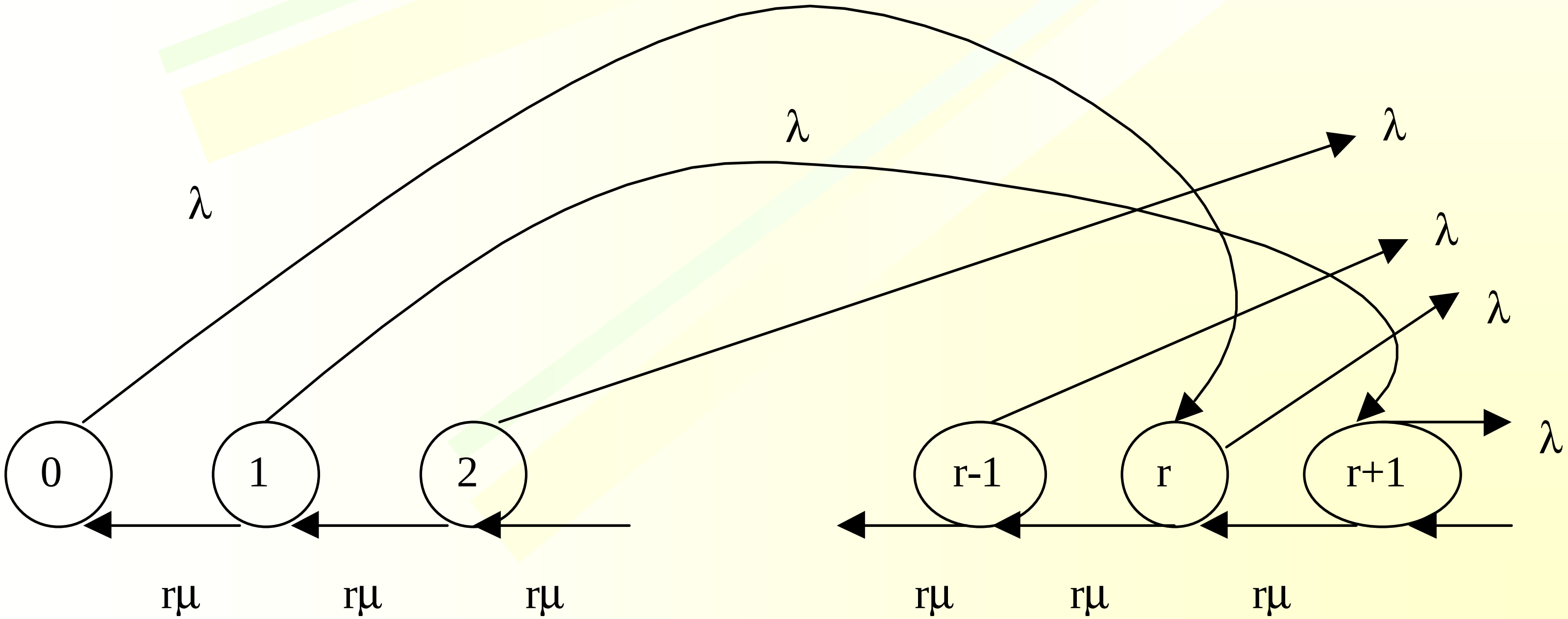
- **M/E_r/1 queue**

Assume r = 4 and the state is 7.
i) How many jobs in the waiting queue? ____



ii) How many more stages the job in the service need to be handled? ____

- **Erlangian service queue: one server of *r* sequential stages**
- **State: number of stages of service to be completed**



M/E$_r$/1 queue

- **A job is processed sequentially through *r* stages, each taking an average of $1/(r\mu)$ time. The total average job processing time is $r \times (1/(r\mu)) = 1/\mu$**

- $$\begin{cases} \lambda \pi_0 = r\mu\pi_1 & \\ (\lambda + r\mu)\pi_k = r\mu\pi_{k+1} & 0 < k < r \\ (\lambda + r\mu)\pi_k = r\mu\pi_{k+1} + \lambda\pi_{k-r} & k \geq r \end{cases}$$

- $$\sum_{k=1}^{r-1}(\lambda + r\mu)\pi_k z^k = \sum_{k=1}^{r-1} r\mu\pi_{k+1} z^k \quad ;(1)$$

$$\sum_{k=r}^{\infty}(\lambda + r\mu)\pi_k z^k = \sum_{k=r}^{\infty} r\mu\pi_{k+1} z^k + \sum_{k=r}^{\infty}\lambda\pi_{k-r} z^k \quad ;(2)$$

$$(1)+(2) \Rightarrow$$

$$(\lambda + r\mu)\sum_{k=1}^{\infty}\pi_k z^k = \sum_{k=1}^{\infty} r\mu\pi_{k+1} z^k + \sum_{k=r}^{\infty}\lambda\pi_{k-r} z^k$$

$$= \frac{r\mu}{z}\sum_{k=1}^{\infty}\pi_{k+1} z^{k+1} + \lambda z^r \sum_{k=r}^{\infty}\pi_{k-r} z^{k-r}$$

M/E$_r$/1 queue (cont'd)

$$(\lambda + r\mu)[\pi^*(z) - \pi_0] = \frac{r\mu}{z}[\pi^*(z) - \pi_1 z - \pi_0] + \lambda z^r \pi^*(z)$$

$$\pi^*(z) = \frac{(\lambda + r\mu)\pi_0 z - r\mu\pi_1 z - r\mu\pi_0}{(\lambda + r\mu)z - r\mu - \lambda z^{r+1}}$$

$$= \frac{(z-1)r\mu\pi_0}{(\lambda + r\mu)z - r\mu - \lambda z^{r+1}} \qquad (\lambda\pi_0 = r\mu\pi_1)$$

$$\pi^*(z) = \frac{r\mu\pi_0}{\dfrac{(\lambda + r\mu)z - r\mu - \lambda z^{r+1}}{z-1}} = \frac{r\mu\pi_0}{r\mu + \dfrac{\lambda z - \lambda z^{r+1}}{z-1}} =$$

$$\frac{r\mu\pi_0}{r\mu - \lambda z\dfrac{1-z^r}{1-z}} = \frac{r\mu\pi_0}{r\mu - \lambda z\sum_{n=0}^{r-1} z^n} = \frac{r\mu\pi_0}{r\mu - \lambda\sum_{n=1}^{r} z^n}$$

M/E$_r$/1 queue (cont'd)

- $F^*(z)\big|_{z=1} = \sum_{k=-\infty}^{\infty} f_k z^k \big|_{z=1} = \sum_{k=-\infty}^{\infty} f_k = 1$ **for f_k to be a PDF**

$$\lim_{z\to 1}\pi^*(z) = \lim_{z\to 1}\frac{r\mu\pi_0}{r\mu - \lambda\sum_{n=1}^{r} z^n} = \frac{r\mu\pi_0}{r\mu - \lambda r} = 1, \pi_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho$$

$$\pi^*(z) = \frac{1-\rho}{1 - \frac{\lambda}{r\mu}\sum_{n=1}^{r} z^n} = \frac{1-\rho}{1 - \frac{\rho}{r}\sum_{n=1}^{r} z^n}$$

- **$E[K]$ = avg no. of stages of service =** $\frac{d}{dz}\pi^*(z)\big|_{z=1} = \frac{(r+1)\rho}{2(1-\rho)}$

$E[C]$ = avg no. of stages left in service = $\sum_{i=1}^{r} i\frac{\rho}{r} = \rho\frac{r+1}{2}$

(i = the stage no. the server is in)

M/E$_r$/1 queue (cont'd)

$$N_q = \frac{E[K]-E[C]}{r} = \frac{\rho^2(r+1)}{2r(1-\rho)}$$

$$N = \rho + N_q = \rho + \frac{\rho^2(r+1)}{2r(1-\rho)}$$

$$T = \frac{N}{\lambda} = \frac{1}{\mu} + \frac{\rho(r+1)}{2r\mu(1-\rho)}$$

$$\pi^*(z) = \frac{1-\rho}{(1-\frac{z}{z_1})(1-\frac{z}{z_2})\ldots(1-\frac{z}{z_r})}$$

$$= (1-\rho)\sum_{n=1}^{r}\frac{A_n}{1-\frac{z}{z_n}}, \quad A_n = \prod_{m=1,m\neq n}^{r}\frac{1}{1-\frac{z_n}{z_m}}$$

$$\pi_k = (1-\rho)\sum_{n=1}^{r}\frac{A_n}{z_n^k}$$

$$(\mathrm{ex})\,\pi^*(z) = \frac{1-\rho}{(1-\frac{z}{z_1})(1-\frac{z}{z_2})(1-\frac{z}{z_3})}$$

$$= (1-\rho)\left(\frac{A_1}{1-\frac{z}{z_1}} + \frac{A_2}{1-\frac{z}{z_2}} + \frac{A_3}{1-\frac{z}{z_3}}\right)$$

$$A_1 = \frac{1}{(1-\frac{z_1}{z_2})(1-\frac{z_1}{z_3})}, A_2 = \frac{1}{(1-\frac{z_2}{z_1})(1-\frac{z_2}{z_3})}, A_3 = \frac{1}{(1-\frac{z_3}{z_1})(1-\frac{z_3}{z_2})}$$

$$A_n = \prod_{m=1,m\neq n}^{r}\frac{1}{1-\frac{z_n}{z_m}}$$

$$\pi_k = (1-\rho)\left(\frac{A_1}{z_1^k} + \frac{A_2}{z_2^k} + \frac{A_3}{z_3^k}\right), \left(\mathrm{Note:} \frac{A}{1-\alpha z} \Leftrightarrow A\alpha^n\right)$$

Non-Markovian systems

- **Many systems are not M/M/x/x/x, having other than Poisson arrival and exponential service time**
- **To ease the analysis of the M/G/1 systems, use the fact that "the average of a sum of r.v.'s is the _____ of their individual average's, regardless of distribution or dependency"**

- **FCFS M/G/1 queue**

  ***W*: waiting time in the queue**

  $= N_q \overline{x}$ **+ waiting time of customer in service**

  (N_q: no. of customers in the queue, $\overline{x}$: avg. service time)

  $$= N_q \overline{x} + (1-\rho)\cdot 0 + \rho \frac{\overline{x^2}}{2\overline{x}}$$

FCFS M/G/1 queue(cont'd)

- $N_q = W\lambda$, $\rho = \lambda \bar{x}$

$$W - \bar{x}\lambda W = \rho \frac{\overline{x^2}}{2\bar{x}}$$ **(Pollaczek-Khinchin eq.)**

$$W = \frac{\lambda \overline{x^2}}{2(1-\rho)}$$

$$T = W + \bar{x},\ N = T\lambda$$

- **M/M/1**

$$B^*(s) = \frac{\mu}{\mu + s}$$

$$\bar{x} = -\frac{d}{ds} B^*(s)\Big|_{s=0} = \frac{1}{\mu},\ \overline{x^2} = \frac{d^2}{ds^2} B^*(s)\Big|_{s=0} = \frac{2}{\mu^2}$$

$$N = \rho + \frac{\lambda^2 \frac{2}{\mu^2}}{2(1-\rho)} = \frac{\rho}{1-\rho}$$

FCFS M/G/1 queue(cont'd)

(Em 6.2) M/E$_r$/1

$$B^*(s) = \left(\frac{r\mu}{r\mu + s}\right)^r$$

$$\bar{x} = -\frac{d}{ds}B^*(s)\bigg|_{s=0} = \frac{1}{\mu}, \overline{x^2} = \frac{d^2}{ds^2}B^*(s)\bigg|_{s=0} = \frac{r+1}{r\mu^2}$$

$$N = \rho + \frac{\dfrac{\lambda^2(1+r)}{r\mu^2}}{2(1-\rho)} = \rho + \frac{\rho^2(1+r)}{2r(1-\rho)}$$

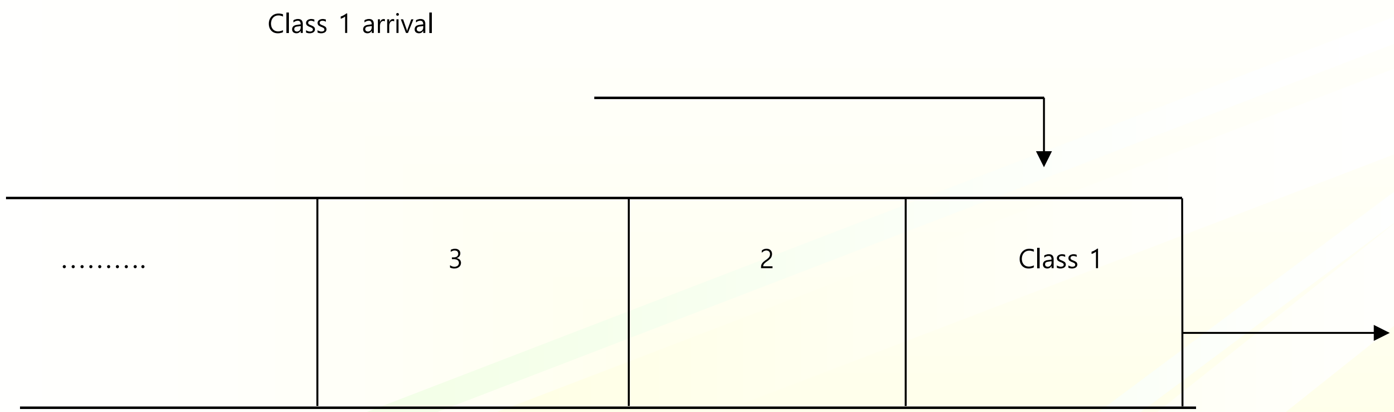
(Em 6.3) M/D/1

$$\bar{x} = C, \quad \overline{x^2} = C^2$$

$$N = \rho + N_q = \lambda C + \frac{\lambda C^2}{2(1-\rho)}\lambda$$

Priority M/G/1 queue

- LCFS (last come first serve) / HOL (head of the line)



- λ_m: arrival rate for class-*m*

  $\overline{x}_m$: average service time for class-*m*

  $\rho_m (= \lambda_m \overline{x}_m)$: fraction of time class-*m* is served
- Highest priority job sees an M/G/1 system (in a preemptive system)
- The next highest sees service available only $(1 - \rho_1)$ of the time
- In a non-preemptive system, the highest priority jobs see an M/G/1 with a ________ service time of the one being served.

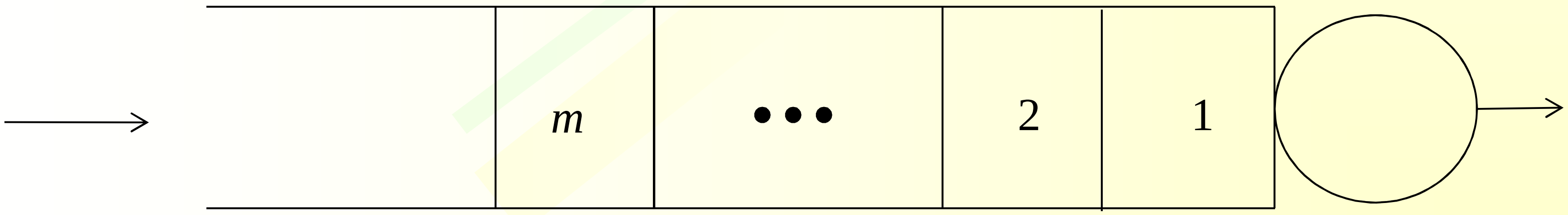
Priority M/G/1 queue(cont'd)

$$w_m = w_0 + \sum_{i=1}^{m} \bar{x}_i (\overbrace{\lambda_i w_i}^{N_i}) + \sum_{i=1}^{m-1} \bar{x}_i (\overbrace{\lambda_i w_m}^{\text{No. of higher priority jobs arriving during } w_m})$$

1st term : delay due to jobs in ________

2nd term: delay due to jobs ________,
which are equal or higher priority

3rd term: delay due to arrivals of ____er
priority jobs while waiting



Priority M/G/1 queue(cont'd)

$$w_1 = w_0 + \rho_1 w_1$$

$$w_1 = \frac{w_0}{1-\rho_1}$$

$$w_2 = w_0 + \rho_1 w_1 + \rho_2 w_2 + \rho_1 w_2$$

$$w_2 = \frac{w_0}{(1-\rho_1-\rho_2)(1-\rho_1)}$$

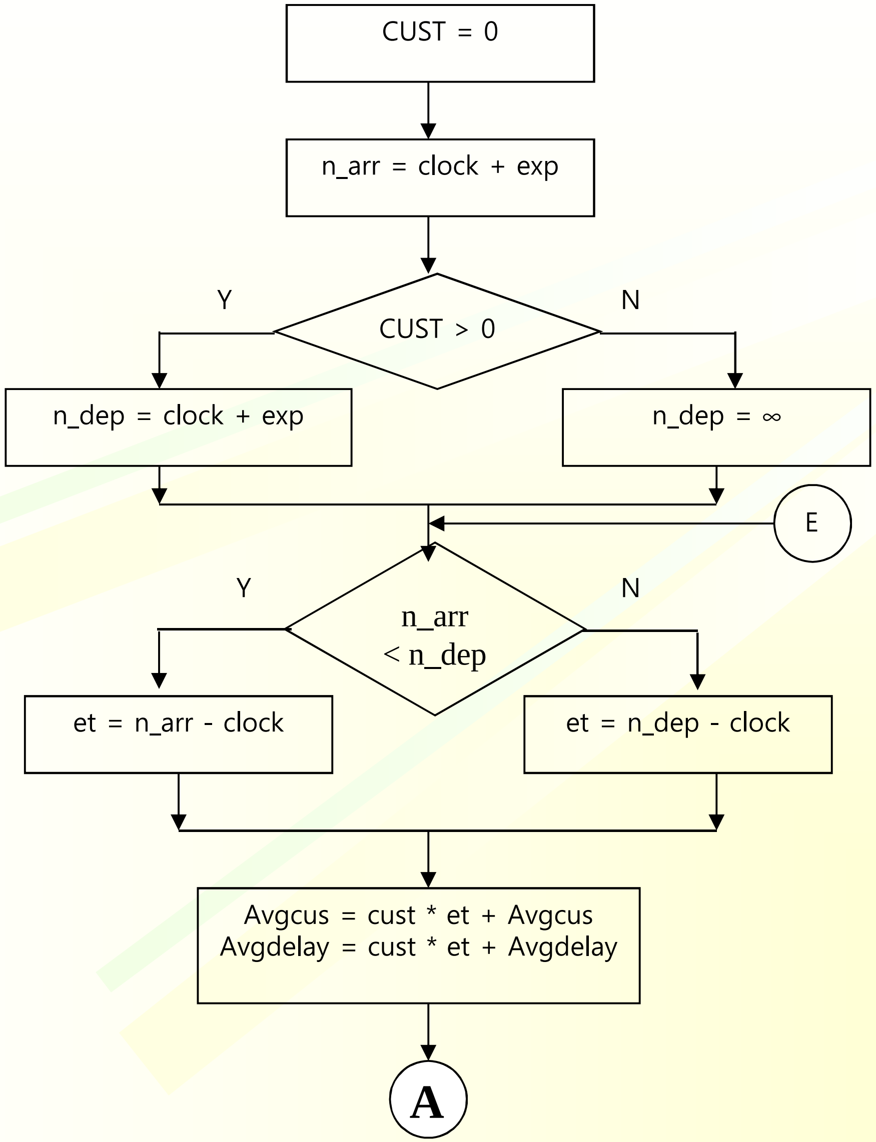
$$\vdots$$

$$w_m = \frac{w_0}{(1-\sum_{i=1}^{m}\rho_i)(1-\sum_{i=1}^{m-1}\rho_i)} = \frac{w_0}{(1-\sigma_m)(1-\sigma_{m-1})}$$

(σ_m : fraction of time spent on classes of equal or ______ priority than class-*m*)

$$w_0 = \begin{cases} \sum_{i=1}^{p}\rho_i \dfrac{\overline{x_i^2}}{2\overline{x}_i} & \text{: for non-preemption} \\ \sum_{i=1}^{m}\rho_i \dfrac{\overline{x_i^2}}{2\overline{x}_i} & \text{: for preemption} \end{cases}$$

Simulation of M/M/1 queue



Simulation of M/M/1 queue (cont'd)

N = Avgcus/clock
T = Avgdelay/t_arr
ρ = busy/clock