

# Lecture 5 : Markov Models

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# Markov Models

- Modeling a system with
  - Continuous state space
  - Discrete state space(chains)
  
- For discrete state space systems
  - $P[S_{n+1} = S_k | S_n = S_i, S_{n-1} = S_j, \dots, S_1 = S_l] = ?$
  
  - Present state  $\neq f(\text{past history})$ 
    - Memoryless
    - Poisson and Bernoulli process
    - Easy to model
  
  - Present state  $= f(\text{past history})$ 
    - Complex process

# Timing of state change

## □ Timing of state change

- Discrete time
- Continuous time

## □ Discrete state space process



- Transition =  $f(\text{Present state})$

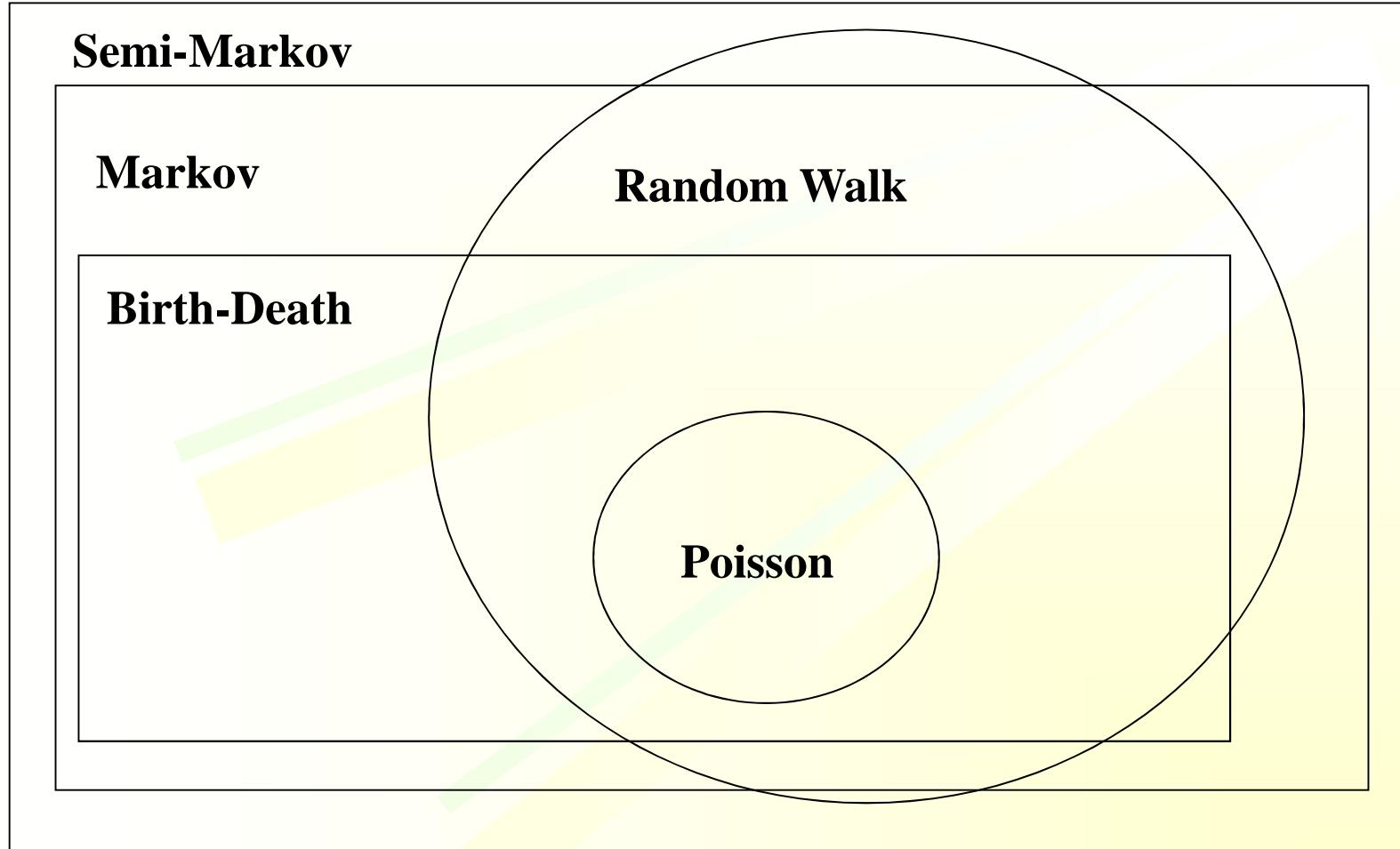
# Discrete state space process (cont'd)

## □ Types

- Based on the time between state change (TBSC) and  $P[S_k/S_j]$
- Semi Markov Process/ Random walk/ Markov Process/ Birth Death Process

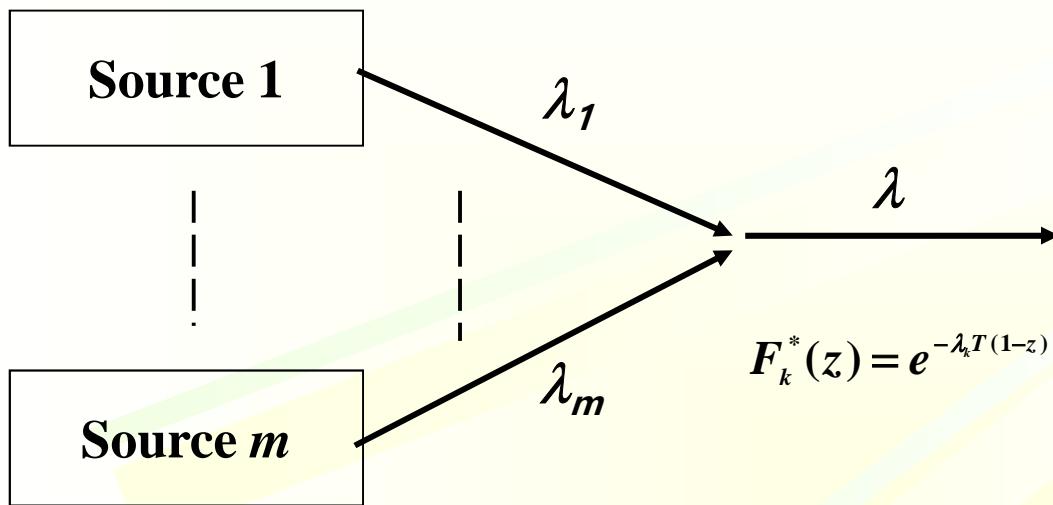
	TBSC	$P[S_k/S_j]$
Semi-Markov	Arbitrary	Arbitrary
Random Walk	Arbitrary	$P_{k-j}$
Markov	_____	_____
Birth-Death	_____	Arbitrary, $ k-j =1$ _____, $ k-j >1$

# Domains of stochastic processes



# Poisson process

- Superposition:  $m$  independent Poisson processes merge together



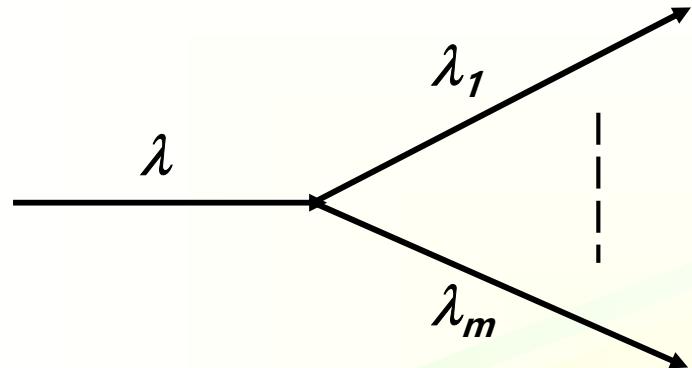
- Total arrivals are (due to \_\_\_\_\_ )

$$F^*(z) = \prod_{k=1}^m F_k^*(z) = e^{-\lambda T(1-z)}$$

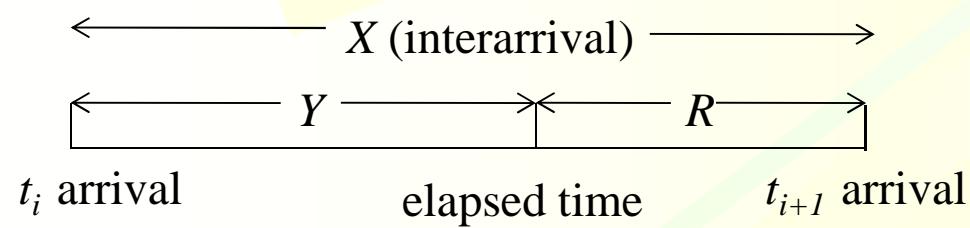
$$\lambda = \sum_{i=1}^m \lambda_i$$

# Poisson process (cont'd)

## □ Decomposition



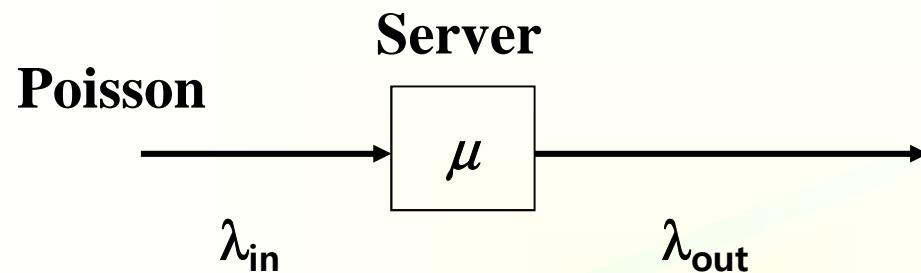
## □ Memoryless



$$\begin{aligned} & P[R \leq r | X > Y] \\ &= \frac{P[R \leq r, X > Y]}{P[X > Y]} = \frac{P[Y < X \leq Y+r]}{P[X > Y]} \\ &= \frac{P[X \leq Y+r] - P[X \leq Y]}{1 - P[X \leq Y]} \\ &= \frac{(1 - e^{-\lambda(Y+r)}) - (1 - e^{-\lambda Y})}{1 - (1 - e^{-\lambda Y})} = \frac{e^{-\lambda Y} - e^{-\lambda(Y+r)}}{e^{-\lambda Y}} \\ &= 1 - e^{-\lambda r} = P[R \leq r] \end{aligned}$$

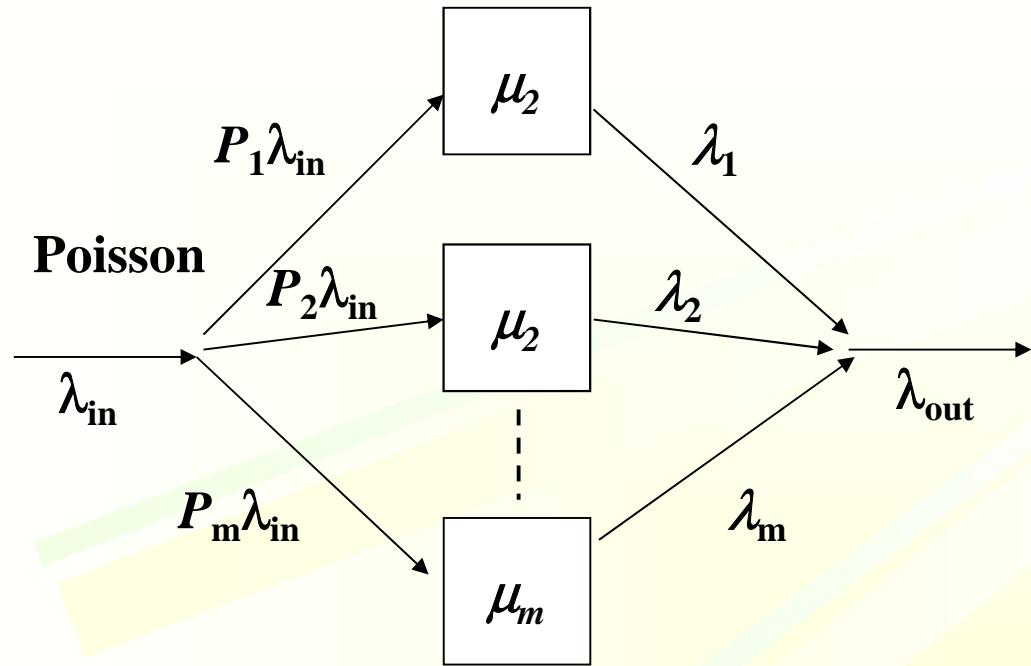
# Poisson process (cont'd)

## □ Invariance property



- Sever: exponentially distributed service time
- If  $\lambda_{\text{in}} < \mu$  then  $\lambda_{\text{out}} = \lambda_{\text{in}}$  and the output is also Poisson

# Poisson process (cont'd)



$$\lambda_{in} = \sum_{i=1}^m P_i \lambda_{in}$$

if  $\forall i \quad P_i \lambda_{in} < \mu_i$

$$\text{then } \lambda_{out} = \sum_{i=1}^m \lambda_i = \sum_{i=1}^m P_i \lambda_{in} = \lambda_{in}$$

# Discrete vs. continuous time Markov chain

## □ Discrete time

- $P[S_k(t_n + \Delta s) | S_j(t_n)] = p_{jk}(t_n)$
- $P(t_n) \equiv [p_{jk}(t_n)]$
- Time spent in  $S_j$  is memoryless → geometric distribution → Bernoulli process

## □ Continuous time

- $P[S_k(t'') | S_j(t')] = h_{jk}(t', t'')$
- $H(t', t'') \equiv [h_{jk}(t', t'')]$
- $(t'' - t')$  is exponentially distributed

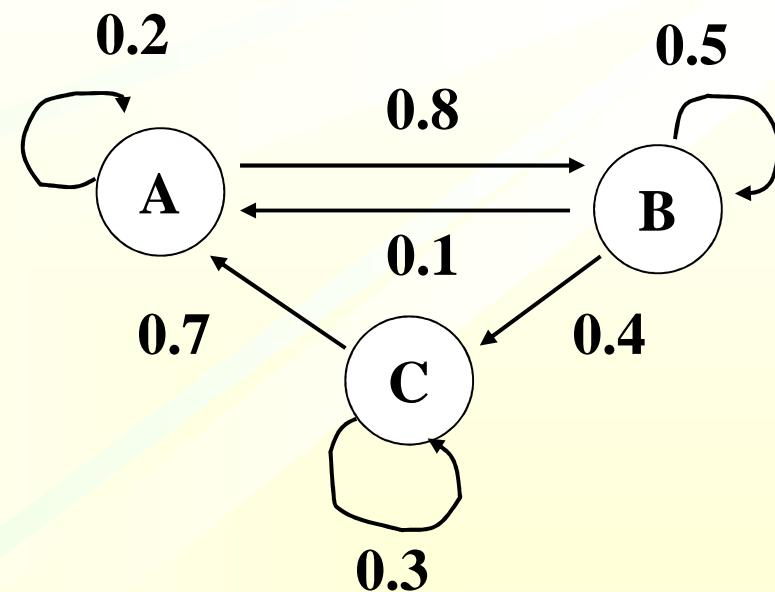
## □ Homogenous Markov system

- $P(t_1) = P(t_2) = P$
- $H(t', t'') = H(0, t'' - t')$  (System parameters do not depend on the absolute value of time, but only on the time difference)

# Representation of Markov chain(MC)

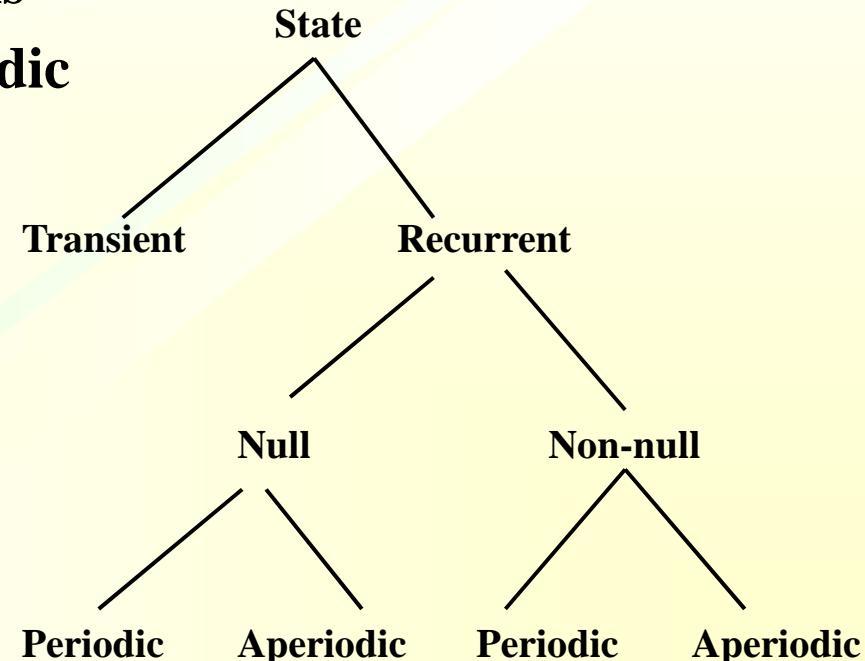
## □ Stochastic matrix / state diagram

<i>States</i>	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.2	0.8	0
<i>B</i>	0.1	0.5	0.4
<i>C</i>	0.7	0	0.3



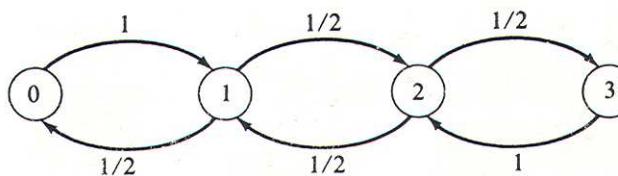
# State classification

- **Recurrent** : once in that state, will return to it with probability 1
- **Transient** : not recurrent
- **Recurrent nonnull** : finite mean time to return
- **Recurrent null** : infinite mean time to return
- **Recurrent aperiodic** : for some  $k$ , return to it in  $k, k+1, \dots, \infty$  transitions
- **Recurrent periodic** : not aperiodic



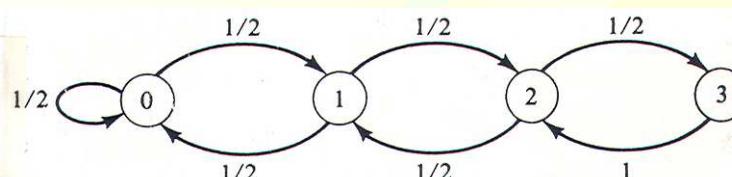
# Example

## □ (Em 5.1)



**FIGURE 5.4** An Example Markov Chain with Periodic States

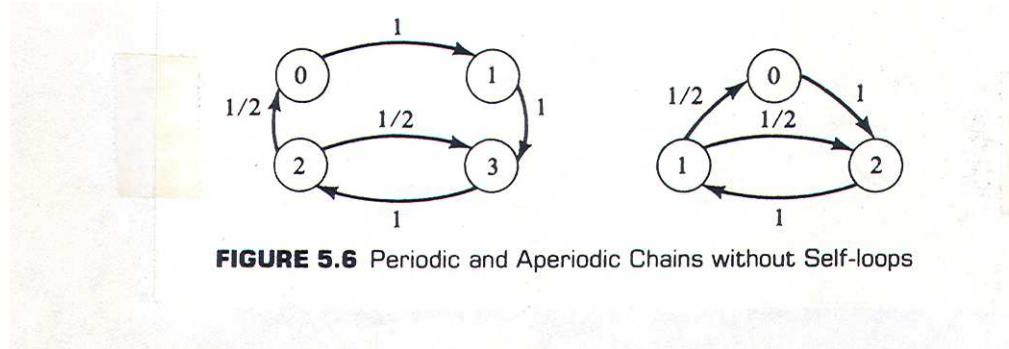
## □ (Em 5.2)



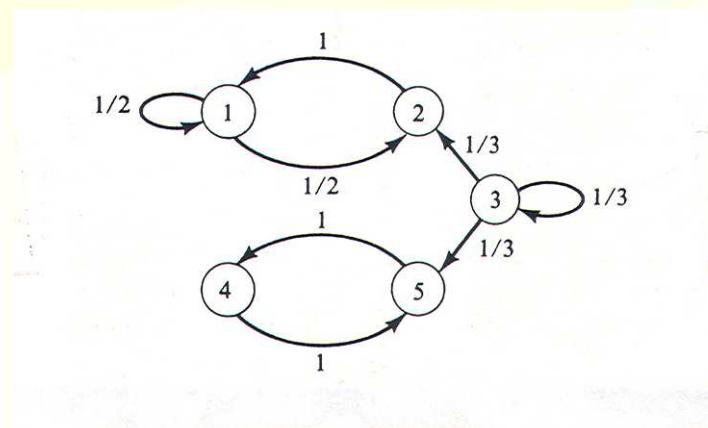
**FIGURE 5.5** An Example Markov Chain with Aperiodic States

# Example (cont'd)

## □ (Em 5.3)



## □ (Em 5.4)



# MC classification

- When *all* state classifications are identical, the MC is said to be of that classification
- *Irreducible*: all states are \_\_\_\_\_ from all other states
- *Theorem* : States of an irreducible MC are all the same types
- *Ergodic* : irreducible, recurrent non null, \_\_\_\_\_.  
An ergodic MC has a limiting probability for each state,  
regardless of \_\_\_\_\_ state

# Discrete time systems

- State probability vector:

$$\vec{P} = [p_0, p_1, \dots, p_{k-1}], \sum_{i=0}^{k-1} p_i = 1 \quad (\text{for an MC of } k \text{ states})$$

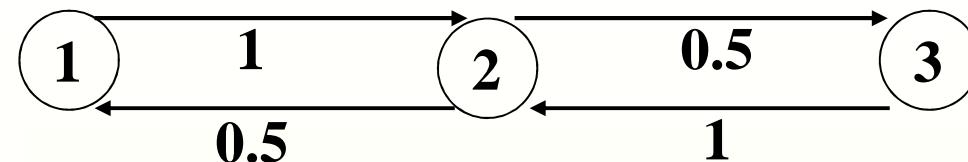
- Single-step transition probability matrix:

$$P = [p_{jk}], \quad \vec{p}(n)P = \vec{p}(n+1)$$

$\vec{p}(n)$ :  $n$ -th step state probability vector

# Discrete time systems (cont'd)

(ex)



$$\vec{p}(0) = [1, 0, 0]$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\vec{p}(0) = [1 \ 0 \ 0]$$

$$\vec{p}(0) \cdot P = [0 \ 1 \ 0] = \vec{p}(1)$$

$$\vec{p}(1) \cdot P = [0.5 \ 0 \ 0.5] = \vec{p}(2)$$

$$\vec{p}(2) \cdot P = [0 \ 1 \ 0] = \vec{p}(3)$$

$$\vec{p}(0) = [0 \ 1 \ 0]$$

$$\vec{p}(0) \cdot P = [0.5 \ 0 \ 0.5] = \vec{p}(1)$$

$$\vec{p}(1) \cdot P = [0 \ 1 \ 0] = \vec{p}(2)$$

$$\vec{p}(2) \cdot P = [0.5 \ 0 \ 0.5] = \vec{p}(3)$$

Period = 2

⋮

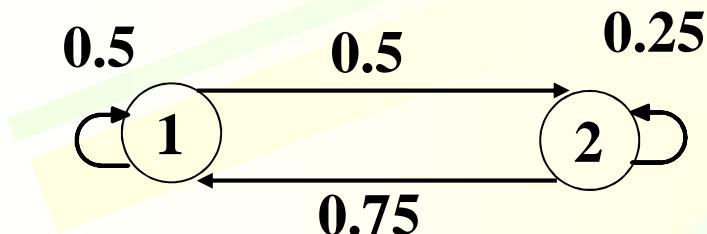
# Multistep state probability

□ Multistep state probability:  $\vec{p}(n) = \vec{p}(0)P^n$

□ For \_\_\_\_\_ system, as a steady state condition

$\vec{p}(\infty) = \vec{p}(0)P^\infty = \pi$  (limiting state probability vector)

□ (Ex)

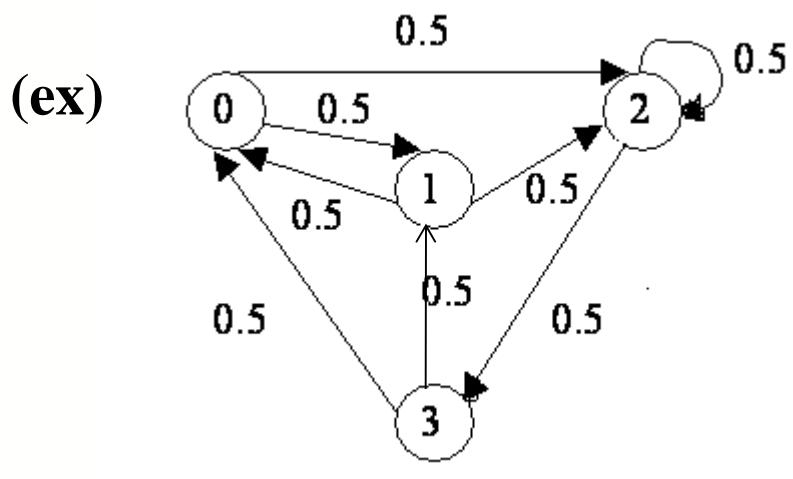


$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.75 & 0.25 \end{bmatrix}, P^3 = \begin{bmatrix} .5938 & .4063 \\ .6094 & .3906 \end{bmatrix}, P^4 = \begin{bmatrix} .6016 & .3984 \\ .5977 & .4023 \end{bmatrix}$$

$$P^6 = \begin{bmatrix} .6001 & .3999 \\ .5999 & .4001 \end{bmatrix}, P^7 = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}, P^{100} = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \Rightarrow \pi = [0.6 \quad 0.4]$$

# Multistep state probability (cont'd)

- Theorem: For a matrix  $A$ , if the sum of each row is 1 and the rows are identical, then  $A = A^2 = \dots = A^\infty$ . For a vector,  $\vec{p}$ , if the sum of the elements is 1, then  $\vec{p}A = [\text{a row vector of } A]$



$$P = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \end{bmatrix}$$

# Multistep state probability (cont'd)

$$\vec{p}(0) = [1 \ 0 \ 0 \ 0]$$

$$\vec{p}(2) = [.25 \ 0 \ .5 \ .25]$$

$$\vec{p}(4) = [.25 \ .1875 \ .375 \ .1875]$$

$$\vec{p}(7) = [.1953 \ .2032 \ .3985 \ .2032] \dots$$

$$P^2 = \begin{bmatrix} .125 & .25 & .375 & .25 \\ .25 & .125 & .375 & .25 \\ .25 & .25 & .375 & .125 \\ .125 & .125 & .5 & .25 \end{bmatrix}$$

$$P^{12} = \begin{bmatrix} .2 & .2001 & .4 & .2 \\ .2001 & .2 & .4 & .2 \\ .2 & .2 & .4 & .2001 \\ .2 & .2001 & .3999 & .2 \end{bmatrix}$$

$$\pi = [.2 \ .2 \ .4 \ .2]$$

$$\vec{p}(1) = [0 \ .5 \ .5 \ 0]$$

$$\vec{p}(3) = [.125 \ .25 \ .375 \ .25]$$

$$\vec{p}(5) = [.1875 \ .2188 \ .4063 \ .1875]$$

$$P^5 = \begin{bmatrix} .2031 & .1875 & .4063 & .2031 \\ .1875 & .2031 & .4063 & .2031 \\ .2031 & .2031 & .3906 & .2031 \\ .2031 & .2031 & .4063 & .1875 \end{bmatrix}$$

$$P^{13} = \begin{bmatrix} .2 & .2 & .4 & .2 \\ .2 & .2 & .4 & .2 \\ .2 & .2 & .4 & .2 \\ .2 & .2 & .4 & .2 \end{bmatrix}$$

# Example

## □ Em 5.3

$$[0.3, 0.3, 0.3, 0.1, 0] \begin{bmatrix} 0.2 & 0.75 & 0.0 & 0.05 & 0.0 \\ 0.3 & 0.30 & 0.3 & 0.10 & 0.0 \\ 0.0 & 0.55 & 0.4 & 0.05 & 0.0 \\ 0.0 & 0.00 & 0.0 & 0.50 & 0.5 \\ 0.6 & 0.00 & 0.0 & 0.00 & 0.4 \end{bmatrix} = [0.15, 0.48, 0.21, 0.11, 0.05]$$

p(0)	p(1)	p(2)	p(3)	p(4)	...	p( $\infty$ )
1	0.20	0.265	0.1805	0.20560	...	0.215554
0	0.75	0.375	0.4350	0.37725	...	0.380389
0	0.00	0.225	0.2025	0.21150	...	0.190194
0	0.05	0.110	0.1170	0.12115	...	0.116653
0	0.00	0.025	0.0650	0.08450	...	0.097210

Table 5.2

p(0)	p(1)	p(2)	p(3)	p(4)	...	p( $\infty$ )
0	0.30	0.15	0.204	0.1974	...	0.215554
1	0.30	0.48	0.372	0.3900	...	0.380389
0	0.30	0.21	0.228	0.2028	...	0.190194
0	0.10	0.11	0.121	0.1193	...	0.116653
0	0.00	0.05	0.075	0.0905	...	0.097210

Table 5.3

# Steady state probability

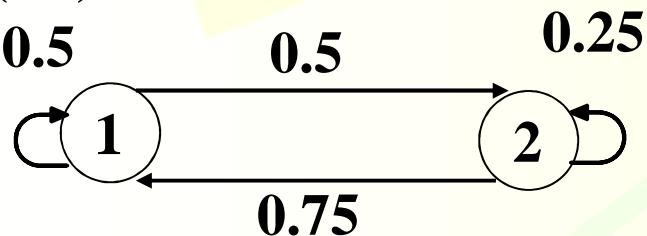
$$\lim_{n \rightarrow \infty} \overset{\rightarrow}{p}(n) = \pi \equiv (\pi_1, \pi_2, \dots, \pi_m)$$

$$\lim_{n \rightarrow \infty} \overset{\rightarrow}{p}(n)P = \lim_{n \rightarrow \infty} \overset{\rightarrow}{p}(n+1)$$

Solve  $\begin{cases} \pi P = \pi \\ \sum_{k=1}^m \pi_k = 1 \text{ in steady state} \end{cases}$

□  $\pi$  is an eigenvector of  $P$  for the eigenvalue 1

□ (Ex)



$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.75 & 0.25 \end{bmatrix}$$

# General multistep transition matrix

$$\vec{p}(n)P = \vec{p}(n+1)$$

$$\vec{p}^*(z) = \sum_{n=0}^{\infty} \vec{p}(n)z^n$$

$$\sum_{n=0}^{\infty} \vec{p}(n)z^n P = \sum_{n=0}^{\infty} \vec{p}(n+1)z^n = \frac{1}{z} \sum_{n=0}^{\infty} \vec{p}(n+1)z^{n+1}$$

$$\vec{p}^*(z)P = \frac{1}{z} \left[ \vec{p}^*(z) - \vec{p}(0) \right]$$

$$z\vec{p}^*(z)P - \vec{p}^*(z) = -\vec{p}(0)$$

$$\vec{p}^*(z)(I - zP) = \vec{p}(0)$$

# General multistep transition matrix (cont'd)

$$\vec{p}^*(z)(I - zP) = \vec{p}(0)$$

$$\overset{\rightarrow}{p}(z) = \overset{\rightarrow}{p}(0)[I - zP]^{-1} \Rightarrow (a)$$

$\overset{\rightarrow}{p}(n) = \overset{\rightarrow}{p}(0)P^n$ , apply Z-transform

$$\overset{\rightarrow}{p}(z) = \overset{\rightarrow}{p}(0)P^*(z) \Rightarrow (b) \text{ (z-transform of } P^n)$$

From (a)&(b),  $P^*(z) = [I - zP]^{-1}$

# General multistep transition matrix (cont'd)

## □ Procedure for obtaining $\vec{p}(n)$

- Obtain  $P^*(z)$  by  $[I-zP]^{-1}$
- Inverse z-transform of  $P^*(z)$  to get  $P^n$
- $\overset{\rightarrow}{p(n)} = \overset{\rightarrow}{p(0)} \overset{\rightarrow}{P^n}$

(ex) 5.10

$$P = \begin{bmatrix} .3 & .2 & .5 \\ .1 & .8 & .1 \\ .4 & .4 & .2 \end{bmatrix} \quad [I - zP] = \begin{bmatrix} 1 - .3z & -.2z & -.5z \\ -.1z & 1 - .8z & -.1z \\ -.4z & -.4z & 1 - .2z \end{bmatrix}$$

# General multistep transition matrix (cont'd)

## □ Matrix inversion

□ minor =  $M_{ij} \equiv$  determinant of matrix excluding  $r_i$  &  $c_j$

□ cofactor =  $C_{ij} \equiv (-1)^{i+j} M_{ij}$

□  $|A| = a_{11} c_{11} + a_{12} c_{12} + a_{13} c_{13}$

□ Adjoint matrix =  $A_{\text{adj}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$

## General multistep transition matrix (cont'd)

$$P = \begin{bmatrix} .3 & .2 & .5 \\ .1 & .8 & .1 \\ .4 & .4 & .2 \end{bmatrix} \quad [I - zP] = \begin{bmatrix} 1 - .3z & -.2z & -.5z \\ -.1z & 1 - .8z & -.1z \\ -.4z & -.4z & 1 - .2z \end{bmatrix}$$

inversion :  $A^{-1} = \frac{A_{\text{adj}}}{|A|}$

$$|I - zP| = (1 - .3z) \begin{vmatrix} 1 - .8z & -.1z \\ -.4z & 1 - .2z \end{vmatrix} + (-.2z)(-1) \begin{vmatrix} -.1z & -.1z \\ -.4z & 1 - .2z \end{vmatrix}$$

$$+ (-.5z) \begin{vmatrix} -.1z & 1 - .8z \\ -.4z & -.4z \end{vmatrix}$$

$$= .1z^3 + .2z^2 - .13z + 1$$

$$= (1 - z)(1 - .5z)(1 + .2z)$$

# General multistep transition matrix (cont'd)

$$\begin{aligned} P^*(z) &= [I - zP]^{-1} \\ &= \frac{[I - zP]_{adj}}{(1-z)(1-.5z)(1+.2z)} \\ &= \frac{\begin{bmatrix} 1-z-\frac{3z^2}{25} & \frac{z}{5}+\frac{4z^2}{25} & \frac{z}{2}-\frac{19z^2}{50} \\ \frac{z}{10}+\frac{z^2}{50} & 1-\frac{z}{2}+\frac{7z^2}{50} & \frac{z}{10}+\frac{z^2}{50} \\ \frac{2z}{5}-\frac{7z^2}{25} & \frac{2z}{5}-\frac{z^2}{25} & 1-\frac{11z}{10}+\frac{11z^2}{50} \end{bmatrix}}{(1-z)(1-.5z)(1+.2z)} \\ &= \frac{1}{1-z}[A]+\frac{1}{1-.5z}[B]+\frac{1}{1+.2z}[C] \end{aligned}$$

## General multistep transition matrix (cont'd)

- Multiply both sides by  $(1-z)$ , and plug 1 for  $z$ , then

$$a_{11} = \frac{1 - 1 + .12}{(1 - .5)(1 + .2)} = \frac{0.12}{0.6} = \frac{1}{5}$$

$$a_{21} = \frac{.1 + .02}{0.6} = \frac{0.12}{0.6} = \frac{1}{5}, \dots$$

# General multistep transition matrix (cont'd)

$$P^*(z) = \frac{1}{1-z}[A] + \frac{1}{1-0.5z}[B] + \frac{1}{1+0.2z}[C]$$

$$= \frac{1}{1-z} \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix} - \frac{1}{1-0.5z} \begin{bmatrix} \frac{-13}{35} & \frac{26}{35} & \frac{-13}{35} \\ \frac{7}{35} & \frac{-14}{35} & \frac{7}{35} \\ \frac{-8}{35} & \frac{16}{35} & \frac{-8}{35} \end{bmatrix} + \frac{1}{1+0.2z} \begin{bmatrix} \frac{3}{7} & \frac{1}{7} & \frac{-4}{7} \\ 0 & 0 & 0 \\ \frac{-3}{7} & \frac{-1}{7} & \frac{4}{7} \end{bmatrix}$$

$$Z[\alpha^n] = \frac{1}{1-\alpha z}$$

# General multistep transition matrix (cont'd)

$$P^n = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix} - (0.5)^n \begin{bmatrix} \frac{-13}{35} & \frac{26}{35} & \frac{-13}{35} \\ \frac{7}{35} & \frac{-14}{35} & \frac{7}{35} \\ \frac{8}{35} & \frac{16}{35} & \frac{-8}{35} \end{bmatrix} + (-0.2)^n \begin{bmatrix} \frac{3}{7} & \frac{1}{7} & \frac{4}{7} \\ 0 & 0 & 0 \\ \frac{3}{7} & \frac{1}{7} & \frac{4}{7} \end{bmatrix}$$

$$P^n = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix} \quad \text{as } n \rightarrow \infty$$

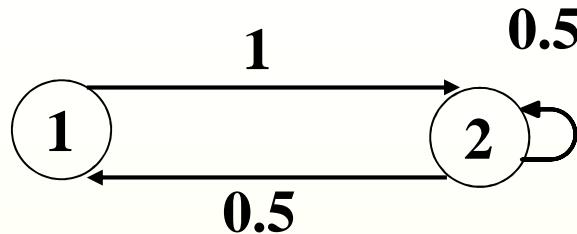
# General multistep transition matrix (cont'd)

- Property of characteristic equation of  $|I - zP|$ 
  - If there is more than one root of magnitude 1, then periodic
  - If the degree is less than the size of row (number of states), then reducible

# General multistep transition matrix (cont'd)

□ Ex. (5.4)

$$P = \begin{bmatrix} 0 & 1 \\ .5 & .5 \end{bmatrix}$$



$$[I-zP] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - z \begin{bmatrix} 0 & 1 \\ .5 & .5 \end{bmatrix} = \begin{bmatrix} 1 & -z \\ .5z & 1-.5z \end{bmatrix} \quad |I-zP| = (1-.5z) - (.5z^2) = 1 - .5z - .5z^2 = (1-z)(1+.5z)$$

$$[I-zP]_{\text{adj}} = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} 1-.5z & z \\ .5z & 1 \end{bmatrix} \quad P^*(z) = \frac{[I-zP]_{\text{adj}}}{(1-z)(1+.5z)} = \frac{\begin{bmatrix} 1-.5z & z \\ .5z & 1 \end{bmatrix}}{(1-z)(1+.5z)} = \frac{[a]}{1-z} + \frac{[b]}{1+.5z}$$

$$[a] = \frac{\begin{bmatrix} 0.5 & 1 \\ 0.5 & 1 \end{bmatrix}}{1+.5} = \frac{\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}}{3}, [b] = \frac{\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}}{1-(-2)} = \frac{\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}}{2} \quad P^*(z) = \frac{1}{1-z} \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} + \frac{1}{1+.5z} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$$

$$\text{Since } A\alpha^n \Leftrightarrow \frac{A}{1-\alpha z}, P^n = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} + (-.5)^n \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

# Mean first passage and recurrence time

- $f_{jk}(n)$ : prob. of the first passage to State- $k$  from State- $j$  in step  $n$
- $p_{jk}(n)$ : prob. of the passage to State- $k$  from State- $j$  in step  $n$

$$p_{jk}(0) = 0 \quad j \neq k$$

$$p_{jj}(0) = 1$$

$$p_{jk}(n) = \sum_{m=1}^n f_{jk}(m) p_{kk}(n-m)$$

$$= \sum_{m=1}^{n-1} f_{jk}(m) p_{kk}(n-m) + f_{jk}(n) p_{kk}(0)$$

$$f_{jk}(n) = p_{jk}(n) - \sum_{m=1}^{n-1} f_{jk}(m) p_{kk}(n-m)$$

$$f_{jk}(1) = p_{jk}(1)$$

$$f_{jk}(2) = p_{jk}(2) - f_{jk}(1) p_{kk}(1)$$

$$f_{jk}(3) = p_{jk}(3) - f_{jk}(1) p_{kk}(2) - f_{jk}(2) p_{kk}(1)$$

⋮

# Mean recurrence time

$$p_{jk}^*(z) = f_{jk}^*(z) p_{kk}^*(z), j \neq k$$

$$p_{jj}^*(z) = 1 + f_{jj}^*(z) p_{jj}^*(z), j = k$$

$$p_{jj}^*(z) = \frac{\Pi_j}{1-z} + g^*(z)$$

$$f_{jj}^*(z) = 1 - \frac{1-z}{\Pi_j + (1-z)g^*(z)}$$

$$\frac{d}{dz} f_{jj}^*(z) \Big|_{z=1} = \left( \frac{1}{\Pi_j + (1-z)g^*(z)} + (1-z)( ) \right) \Big|_{z=1} = \frac{1}{\Pi_j}$$

$$m_{jj} = \frac{1}{\Pi_j} : \text{mean recurrence time}$$

# Example

## □ (Em 5.11)

Main A

Call B

Call C

End

Subr B

Return

Subr C

If ( ) call B

Return

FIGURE 5.8 A Modular Program

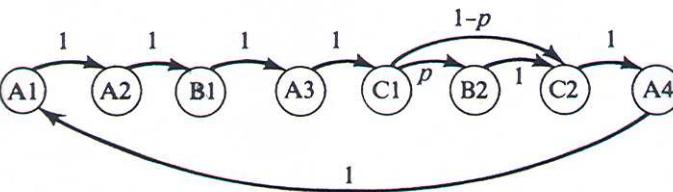


FIGURE 5.9 Markov Chain for a Modular Program

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 1-p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$[\pi_1 \dots \pi_8]P = [\pi_1 \dots \pi_8]$$



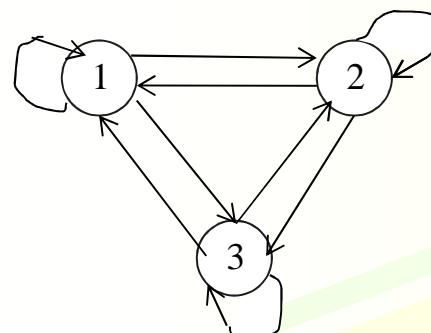
$$\pi_8 = \pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5, p\pi_5 = \pi_6, (1-p)\pi_5 + \pi_6 = \pi_7, \pi_7 = \pi_8$$

$$(1-p)\pi_1 + p\pi_1 = \pi_7, \pi_7 = \pi_1;$$

$$7\pi_1 + p\pi_1 = 1; \pi_1 = 1/(7+p); m_{11} = 7 + p$$

# Mean first passage time

$$m_{jk} = p_{jk} \times 1 + \sum_{n \neq k} p_{jn}(1 + m_{nk})$$



$$m_{13} = p_{13} + p_{11}(1 + m_{13}) + p_{12}(1 + m_{23})$$

$$m_{23} = p_{23} + p_{21}(1 + m_{13}) + p_{22}(1 + m_{23})$$

$$m_{13} = (p_{13} + p_{12}(1 + m_{23}) + p_{11}) / (1 - p_{11})$$

$$m_{23} = (p_{23} + p_{21}(1 + m_{13}) + p_{22}) / (1 - p_{22})$$

$$m_{13} = \frac{\frac{1}{3} + \frac{1}{3}(1 + m_{23}) + \frac{1}{3}}{\frac{2}{3}}; m_{23} = \frac{\frac{1}{3} + \frac{1}{3}(1 + m_{13}) + \frac{1}{3}}{\frac{2}{3}}$$

$$m_{13} = \frac{1 + 1 + m_{23} + 1}{2} = \frac{3 + m_{23}}{2}; m_{23} = \frac{1 + 1 + m_{13} + 1}{2} = \frac{3 + m_{13}}{2}$$

$$m_{13} = \frac{3 + m_{23}}{2} = \frac{3 + \frac{3 + m_{13}}{2}}{2} = \frac{9 + m_{13}}{4}; m_{13} = 3; m_{23} = 3$$

# Continuous time systems

- $H(t', t'') \equiv [h_{jk}(t', t'')] = P[S_k(t'') | S_j(t')], t'' = t' + \Delta t$
- Assume  $\Delta t$  is small enough such that only one transition is acceptable. Then

$$\vec{p}(t)P(t, t+\Delta t) = \vec{p}(t + \Delta t)$$

$$\vec{p}(t)P(t, t+\Delta t) - \vec{p}(t) = \vec{p}(t + \Delta t) - \vec{p}(t)$$

$$\vec{p}(t) \frac{P(t + \Delta t) - I}{\Delta t} = \frac{\vec{p}(t + \Delta t) - \vec{p}(t)}{\Delta t}$$

$$Q(t) = \lim_{\Delta t \rightarrow 0} \frac{P(t, t + \Delta t) - I}{\Delta t}$$

$$\vec{p}(t)Q(t) = \frac{d}{dt} \vec{p}(t)$$

# Transition rate matrix

- Instantaneous rate of change of probabilities

$$q_{jk}(t) = \lim_{\Delta t \rightarrow 0} \frac{p_{jk}(t, t + \Delta t)}{\Delta t}, j \neq k$$

- $q_{jk}(t)$  is the PDF of the time between transitions from State- $j$  to State- $k$  ( $j \neq k$ ), while the transitions are memoryless (MC).
- Thus they are the parameters of \_\_\_\_\_ distribution function
- $q_{jk}(t)$  ( $j = k$ ) is equal to the sum of all the rates for changing from the current state to all other states  $\times (-1)$
- $q_{kk}(t)$  is the PDF rate for staying in State- $k$ .  $1 - q_{kk}(t)$  is the rate for leaving State- $k$ . Subtracting I results in the negative sign.

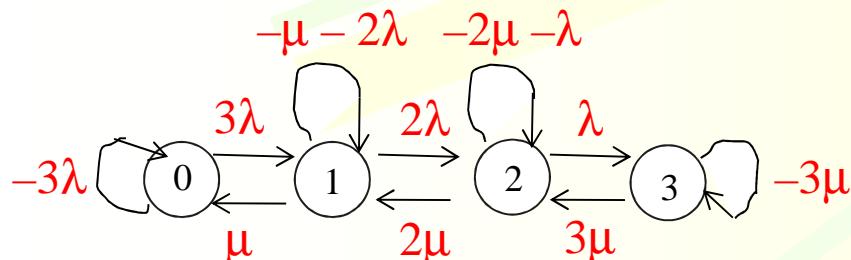
# Steady state probability

- $Q(t)$  is the transition rate matrix, and the rates are constant with  $t$  for homogenous MC.

$$\vec{p}(t)Q = \frac{d}{dt} \vec{p}(t)$$

$$P[S_k | S_j] = \frac{q_{jk}}{\sum_{l \neq j} q_{jl}}$$

- (Em 5.12) Three people sharing three phones.



$$Q = \begin{bmatrix} -3\lambda & 3\lambda & 0 & 0 \\ \mu & -\mu - 2\lambda & 2\lambda & 0 \\ 0 & 2\mu & -2\mu - \lambda & \lambda \\ 0 & 0 & 3\mu & -3\mu \end{bmatrix}$$

$$\pi_0 = \frac{1}{1 + 3\frac{\lambda}{\mu} + 3\frac{\lambda^2}{\mu^2} + \frac{\lambda^3}{\mu^3}}, \pi_1 = \frac{\frac{3\lambda}{\mu}}{1 + 3\frac{\lambda}{\mu} + 3\frac{\lambda^2}{\mu^2} + \frac{\lambda^3}{\mu^3}}$$

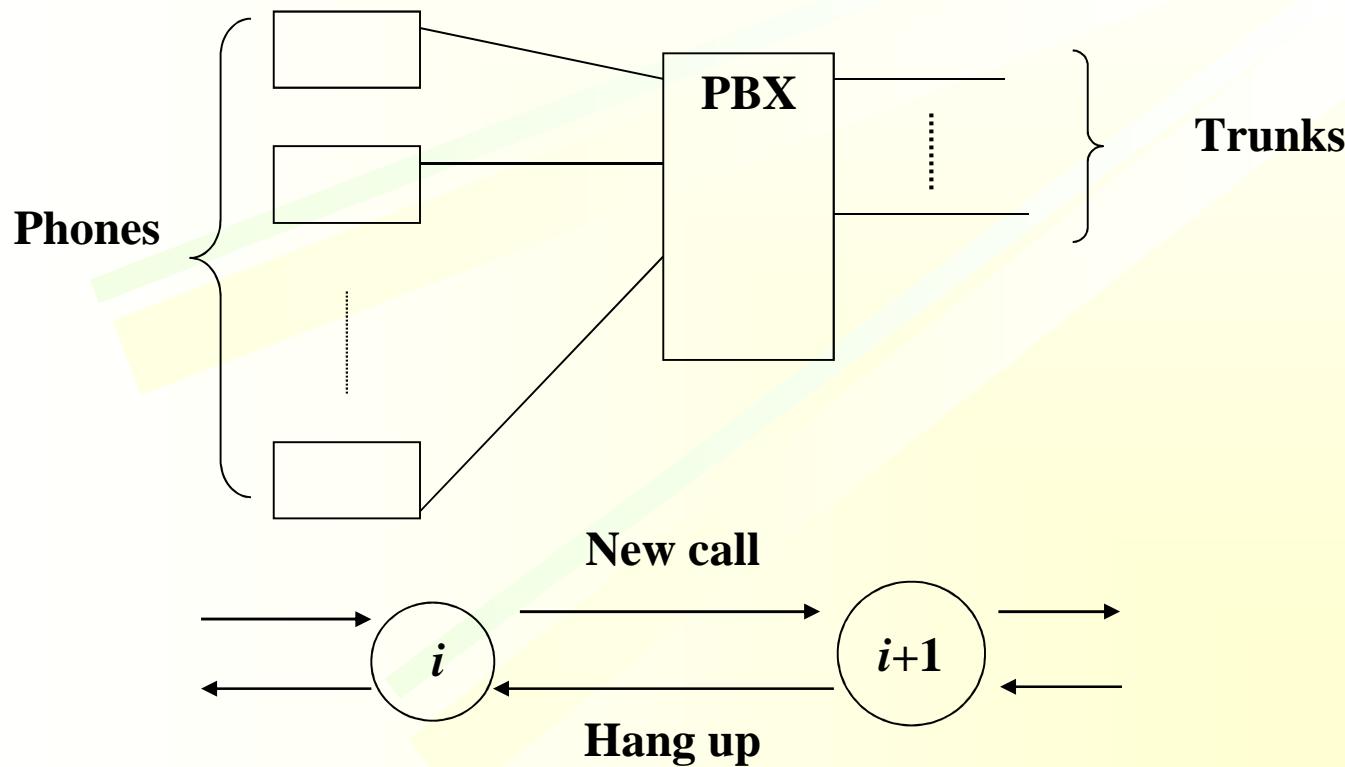
$$\pi_2 = \frac{\frac{3\lambda^2}{\mu^2}}{1 + 3\frac{\lambda}{\mu} + 3\frac{\lambda^2}{\mu^2} + \frac{\lambda^3}{\mu^3}}, \pi_3 = \frac{\frac{\lambda^3}{\mu^3}}{1 + 3\frac{\lambda}{\mu} + 3\frac{\lambda^2}{\mu^2} + \frac{\lambda^3}{\mu^3}}$$

- Steady state probabilities

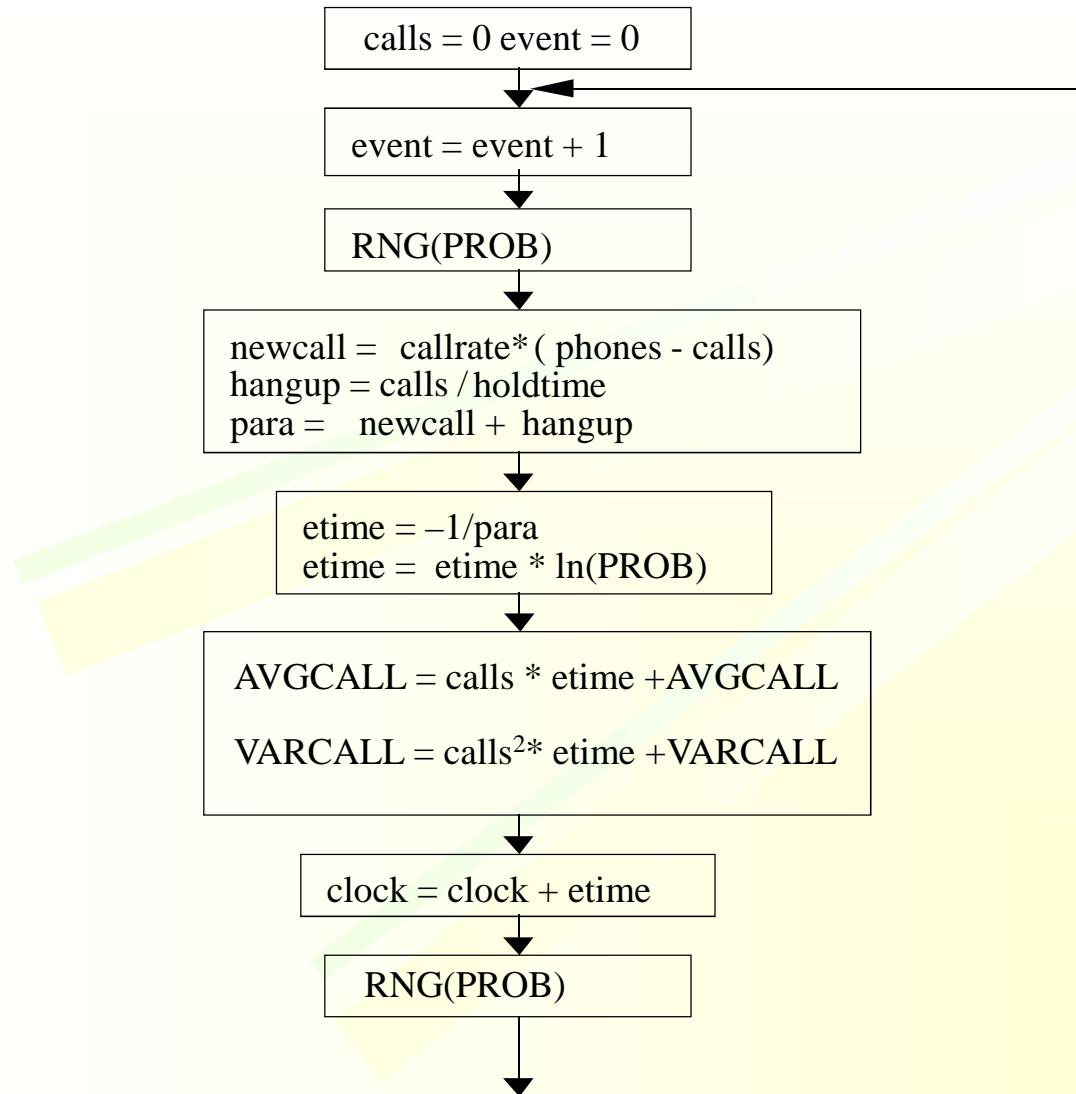
$$\pi = \lim_{t \rightarrow \infty} \vec{p}(t), \pi Q = \vec{0}, \sum_i \pi_i = 1$$

# Simulation of Markovian systems

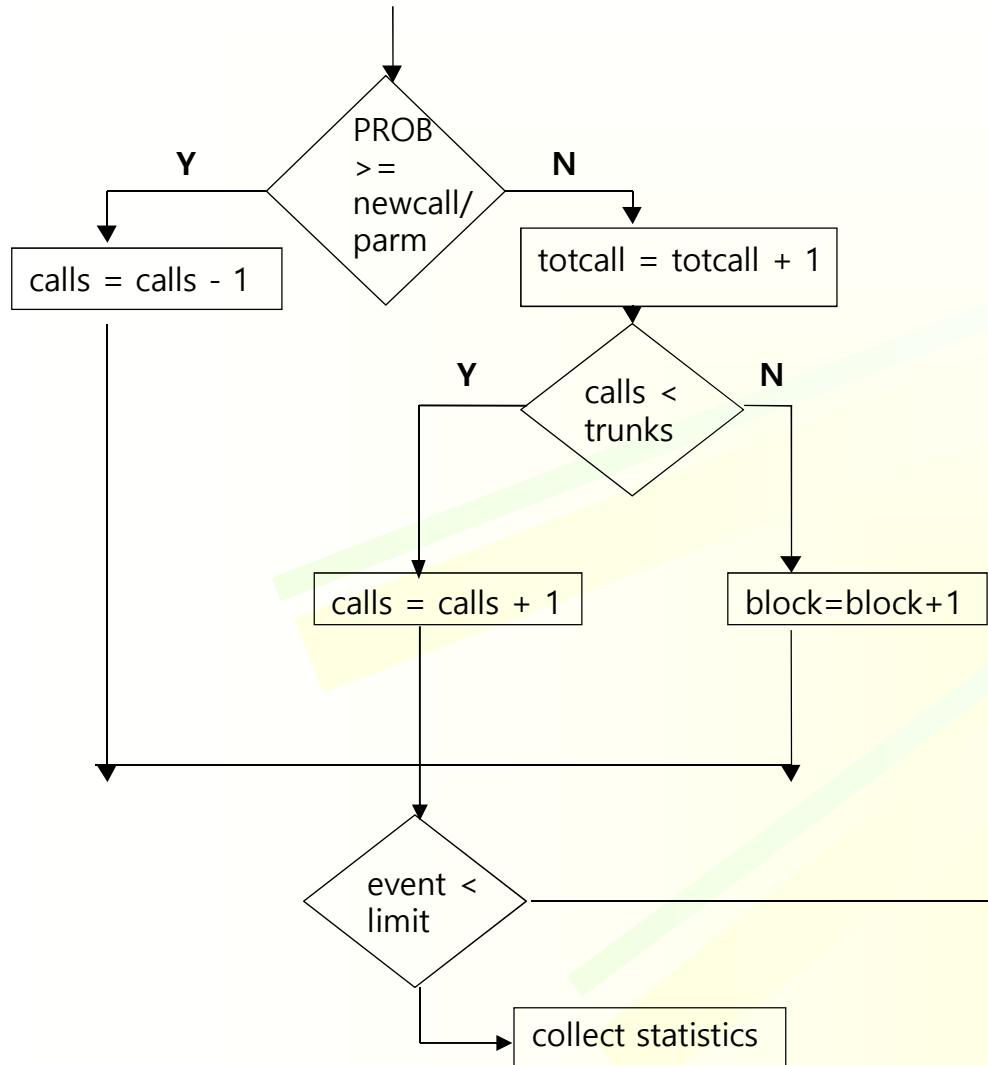
- Event-driven simulation
  - (ex)



# Simulation of Markovian systems (cont'd)



# Simulation of Markovian systems (cont'd)



$$AVGCALL = \frac{AVGCALL}{clock}$$

$$VARCALL = \sqrt{\frac{VARCALL}{clock} - AVGCALL^2}$$

$$BLOCK = \frac{block}{totcall}$$