

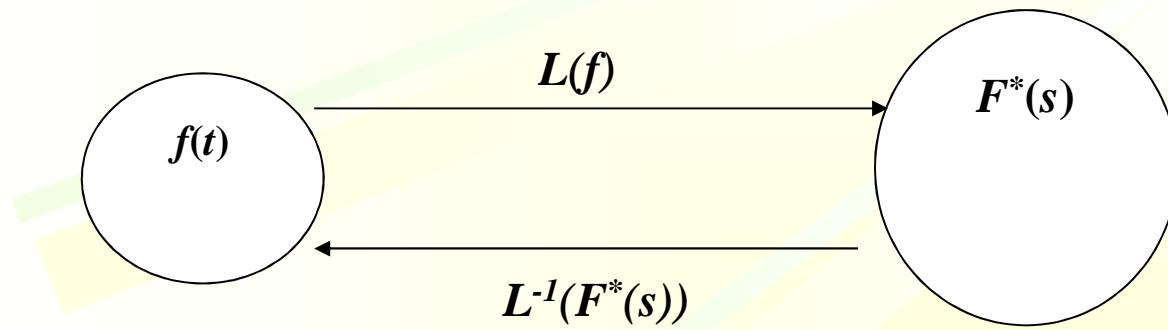
# Lecture 3 : *Transform Theory*

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# Transform

- A \_\_\_\_\_ defined on a domain of functions of  $t$  which produces a new function of  $s$
- For easily solving equations



# Transform (Cont'd)

## □ L: linear operator

$$L(\alpha f(t) + \beta g(t)) = \alpha L(f(t)) + \beta L(g(t))$$

## □ Applications

Transform	Application
Fourier	Sine & cosine Equation
Laplace	Exponential & differential Eq.
Z	Geometric series & difference Eq.

# Z-Transform

- For discrete-valued functions

- $Z(f) = F^*(z) \equiv \sum_{k=-\infty}^{\infty} f_k z^k, z \neq 0$

$f_k = 0$  ( $k < 0$ ) for probability density

Thus  $F^*(z) \equiv \sum_{k=0}^{\infty} f_k z^k$

- (ex) Z-transform of PDF of geometric distribution

$$\begin{aligned} F^*(z) &= \sum_{k=1}^{\infty} (1-p)^{k-1} p z^k \\ &= \frac{p}{1-p} \sum_{k=1}^{\infty} ((1-p)z)^k \\ &= \frac{p}{1-p} \left( \frac{1 - 1 + (1-p)z}{1 - (1-p)z} - 1 \right) \\ &= \frac{p}{1-p} \left( \frac{1 - (1-p)z}{1 - (1-p)z} \right) \\ &= \frac{pz}{1 - (1-p)z} \end{aligned}$$

# Z-transform Table

## Z-Transforms

Function	$\Leftrightarrow$	Transform
$\delta_n$	$\Leftrightarrow$	1
$u_n$	$\Leftrightarrow$	$\frac{1}{1 - z}$
$A\alpha^n$	$\Leftrightarrow$	$\frac{A}{1 - \alpha z}$
$n$	$\Leftrightarrow$	$\frac{z}{(1 - z)^2}$
$\binom{n+m-1}{m-1} \alpha^n$	$\Leftrightarrow$	$\frac{1}{(1 - \alpha z)^m}$
$\frac{1}{n!}$	$\Leftrightarrow$	$e^z$
$f_{n+k} \quad k > 0$	$\Leftrightarrow$	$\frac{F(z)}{z^k} - \sum_{n=1}^k z^{n-k-1} f_{n-1}$
$f_{n-k} \quad k > 0$	$\Leftrightarrow$	$z^k F(z)$
$f_{n/k} \quad n = 0, k, 2k, \dots$	$\Leftrightarrow$	$F(z^k)$
$nf_n$	$\Leftrightarrow$	$z \frac{d}{dz} F(z)$
$\frac{n!}{(n-m)!} f_n$	$\Leftrightarrow$	$z^m \frac{d^m}{dz^m} F(z)$

Note:  $\delta_n$  is the unit function which is 1 for  $n = 0$  and 0 for  $n \neq 0$  and  $u_0$  is the unit step function which is 1 for  $n \geq 0$  and 0 for  $n < 0$ .

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# Z-Transform (Exercise)

□ Obtain  $F^*(z)$  if  $f_k = k + k^2$

$$f_k = k : \sum_{k=0}^{\infty} kz^k = \sum_{k=0}^{\infty} \left( \frac{d}{dz} z^k \right) z = z \frac{d}{dz} \left( \frac{1}{1-z} \right) = z \frac{1}{(1-z)^2}$$

$$\begin{aligned} f_k = k^2 : \sum_{k=0}^{\infty} k^2 z^k &= \sum_{k=0}^{\infty} z \frac{d}{dz} \left( z \frac{d}{dz} z^k \right) = \sum_{k=0}^{\infty} z \left( \frac{d}{dz} z^k + z \frac{d^2}{dz^2} z^k \right) \\ &= z \frac{d}{dz} \sum_{k=0}^{\infty} z^k + z^2 \frac{d^2}{dz^2} \sum_{k=0}^{\infty} z^k \\ &= z \left( \frac{1}{1-z} \right)' + z^2 \left( \frac{1}{1-z} \right)'' = z \frac{1}{(1-z)^2} + z^2 \left( \frac{1}{(1-z)^2} \right)' \end{aligned}$$

$$\begin{aligned} &= \frac{z}{(1-z)^2} + \frac{(-2)(1-z)(-1)}{(1-z)^4} = \frac{z(1-z) + 2z^2}{(1-z)^3} \\ &= \frac{z + z^2}{(1-z)^3} \end{aligned}$$

$$F^*(z) = \frac{z}{(1-z)^2} + \frac{z + z^2}{(1-z)^3} = \frac{z - z^2 + z + z^2}{(1-z)^3} = \frac{2z}{(1-z)^3}$$

## Z-Transform (Exercise)

□ Obtain the  $z$ -transform of the Poisson density function,

$$f_k = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$\begin{aligned} F^*(z) &= \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} z^k = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t z)^k}{k!} \quad (\text{Note: } \sum_{k=0}^{\infty} \frac{a^k}{k!} = \underline{\hspace{2cm}}) \\ &= e^{-\lambda t} e^{\lambda t z} = e^{-\lambda t(1-z)} \end{aligned}$$

# Inverse Z-transform

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$$\square f_n = \frac{1}{n!} \frac{d^n}{dz^n} F^*(z) \Big|_{z=0}$$

- Table look-up technique
- Partial expansion

## Inverse Z-transform (Exercise)

$$F^*(z) = \frac{9-11z}{3-10z+3z^2}$$

$$F^*(z) = \frac{A}{1-3z} + \frac{B}{3-z}$$

(Method 1) Solve linear system

$$3A - Az + B - 3Bz = (3A + B) - (A + 3B)z = 9 - 11z$$

$$3A + B = 9 \quad (1), \quad A + 3B = 11 \quad (2); \quad (1) \times 3 - (2) \Rightarrow 8A = 16 \therefore A = 2, B = 3$$

$$(Method 2) Partial expansion: \frac{9-11z}{(1-3z)(3-z)} = \frac{A}{1-3z} + \frac{B}{3-z}$$

$$Multiply both sides by (1-3z), then we have \frac{9-11z}{(3-z)} = A + \frac{B}{3-z}(1-3z)$$

$$Next \ plug \frac{1}{3} \ to z, then we have \frac{9-\frac{11}{3}}{3-\frac{1}{3}} = \frac{\frac{16}{3}}{\frac{8}{3}} = 2 = A$$

To get B, multiply both sides by \_\_\_\_\_ and plug \_\_\_ to z. Then we have

$$\frac{9-33}{-8} = 3 = B \quad (Which method do you like?)$$

$$F^*(z) = \frac{2}{1-3z} + \frac{3}{3-z} \quad (Note: A\alpha^n \Leftrightarrow \frac{A}{1-\alpha z})$$

$$f_k = 2 \bullet 3^k + 1 \bullet \left(\frac{1}{3}\right)^k$$

# Inverse Z-transform (Exercise: same order)

$$F^*(z) = \frac{4 - \frac{3}{2}z - \frac{1}{2}z^2}{12 - 13z + 3z^2} = -\frac{1}{6} + \frac{6 - \frac{11}{3}z}{12 - 13z + 3z^2} = -\frac{1}{6} + \frac{A}{3-z} + \frac{B}{4-3z}$$

(Method 1) Solve linear system

$$4A + 3B = 6 \quad (1), \quad -3A - B = -11/3 \quad (2); \quad (2) \times 3 + (1) \Rightarrow -5A = -5 \therefore A = 1, B = 2/3$$

(Method 2) Partial expansion:  $\frac{6 - (11/3)z}{(3-z)(4-3z)} = \frac{A}{3-z} + \frac{B}{4-3z}$

Multiply both sides by  $(3-z)$ , then we have  $\frac{6 - (11/3)z}{(4-3z)} = A + \frac{B}{4-3z}(3-z)$

Next plug 3 to  $z$ , then we have  $\frac{6-11}{-5} = 1 = A$

To get  $B$ , multiply both sides by \_\_\_\_\_ and plug \_\_\_ to  $z$ . Then we have

$$\frac{6-44/9}{3-4/3} = \frac{10/9}{5/3} = \frac{2}{3} = B$$

$$F^*(z) = -\frac{1}{6} + \frac{1}{3-z} + \frac{2/3}{4-3z} \quad (\text{Note: } \delta_k \Leftrightarrow 1; A\alpha^n \Leftrightarrow \frac{A}{1-\alpha z})$$

$$f_k = -\frac{1}{6}\delta_k + \frac{1}{3}\left(\frac{1}{3}\right)^k + \frac{1}{6}\left(\frac{3}{4}\right)^k$$

$$f_k = -\frac{1}{6} + \frac{1}{3} + \frac{1}{6} = \frac{1}{3} \quad (k=0); \quad \frac{1}{3}\left(\frac{1}{3}\right)^k + \frac{1}{6}\left(\frac{3}{4}\right)^k \quad (k \geq 1)$$

# Inverse Z-transform (to get only a few first terms)

$$F^*(z) = \frac{4 - \frac{3}{2}z - \frac{1}{2}z^2}{12 - 13z + 3z^2} \rightarrow f_k = \frac{1}{3} (k=0); \quad \frac{1}{3}\left(\frac{1}{3}\right)^k + \frac{1}{6}\left(\frac{3}{4}\right)^k (k \geq 1)$$

$$F^*(z) = f_0 z^0 + f_1 z^1 + f_2 z^2 + \dots = f_0 + f_1 z^1 + f_2 z^2 + \dots$$

$$F^*(0) = f_0; \text{ also } f_k = \frac{1}{n!} \frac{d^n}{dz^n} F^*(z) \Big|_{z=0}$$

$$\begin{array}{r} \frac{1}{3} + \frac{17}{72}z + \dots \\ \hline 12 - 13z + 3z^2 \end{array} \quad \begin{array}{l} 4 - \frac{3}{2}z - \frac{1}{2}z^2 \\ \hline 4 - \frac{13}{3}z + z^2 \end{array} \quad So, f_0 = \underline{\hspace{2cm}}, f_1 = \underline{\hspace{2cm}}$$

$$\begin{array}{r} \frac{17}{6}z - \frac{3}{2}z^2 \dots \\ \hline \frac{17}{6}z - \frac{3}{2}z^2 \dots \end{array}$$

# Inverse Z-transform (Exercise: multiple roots)

$$F^*(z) = \frac{z+z^2}{(1-z)^3} = \frac{A}{1-z} + \frac{B}{(1-z)^2} + \frac{C}{(1-z)^3}$$

$$F^*(z) = f_0 z^0 + f_1 z^1 + f_2 z^2 + \dots = f_0 + f_1 z^1 + f_2 z^2 + \dots$$

$$A(1-z)^2 + B(1-z) + C = A(1-2z+z^2) + B(1-z) + C$$

$$= (A+B+C) + (-2A-B)z + Az^2 = z + z^2; \therefore A=1; -2A-B=1; B=-3; C=2$$

or

Multiply both sides by  $(1-z)^3$  and plug 1 to  $z : 2 = C$

$$\frac{-2+z+z^2}{(1-z)^3} = \frac{-(2+z)(1-z)}{(1-z)^3} = \frac{-(2+z)}{(1-z)^2} = \frac{A}{1-z} + \frac{B}{(1-z)^2}$$

Multiply both sides by  $(1-z)^2$  and plug 1 to  $z : -3 = B$

$$\frac{3-(2+z)}{(1-z)^2} = \frac{1-z}{(1-z)^2} = \frac{A}{1-z} \quad \therefore A=1$$

$$F^*(z) = \frac{z+z^2}{(1-z)^3} = \frac{1}{1-z} + \frac{-3}{(1-z)^2} + \frac{2}{(1-z)^3}, \quad \binom{n+m-1}{m-1} \alpha^n \Leftrightarrow \frac{1}{(1-\alpha z)^m}$$

$$\frac{1}{1-z} \xrightarrow{\alpha=1, m=1} \binom{k}{0} 1^k = 1; \quad \frac{-3}{(1-z)^2} \xrightarrow{\alpha=1, m=2} -3 \binom{k+2-1}{2-1} 1^k = (k+1)(-3) = -3k - 3$$

$$\frac{2}{(1-z)^3} \xrightarrow{\alpha=1, m=3} 2 \binom{k+3-1}{3-1} 1^k = 2 \frac{(k+2)(k+1)}{2} = k^2 + 3k + 2; \text{ So, } f_k = k^2$$

# Difference Equations

□ (ex)  $2f_k = f_{k-1} + f_{k+1}$ ,  $f_0 = \frac{2}{3}$

$$\begin{aligned}\sum_{k=0}^{\infty} 2f_k z^k &= \sum_{k=0}^{\infty} f_{k-1} z^k + \sum_{k=0}^{\infty} f_{k+1} z^k \\ &= z \sum_{k=0}^{\infty} f_{k-1} z^{k-1} + z^{-1} \sum_{k=0}^{\infty} f_{k+1} z^{k+1} \\ &= z(F^*(z) - f_0) + z^{-1}(F^*(z) - f_0)\end{aligned}$$

$$2F^*(z) = zF^*(z) + \frac{1}{z}(F^*(z) - \frac{2}{3})$$

$$F^*(z) = \frac{\frac{-2}{3z}}{2-z-\frac{1}{z}} = \frac{\frac{2}{3}}{1-2z+z^2} = \frac{\frac{2}{3}}{(1-z)^2}$$

$$f_k = \frac{2}{3} \binom{k+2-1}{2-1} = \frac{2}{3}(k+1)$$

$$f_0 = \frac{2}{3}, f_1 = \frac{4}{3}, f_2 = \frac{6}{3}, \dots$$

# Difference Equations (Exercise)

$$f_{k+1} - f_k = 2k + 1$$

$$f_{k+1}z^k - f_k z^k = 2kz^k + z^k$$

$$\frac{1}{z} \sum_{k=0}^{\infty} f_{k+1}z^{k+1} - \sum_{k=0}^{\infty} f_k z^k = 2 \sum_{k=0}^{\infty} kz^k + \sum_{k=0}^{\infty} z^k$$

$$\sum_{k=0}^{\infty} kz^k = z + 2z^2 + 3z^3 + 4z^4 + \dots (= A)$$

$$- z^2 + 2z^3 + 3z^4 + \dots (= zA)$$

$$\frac{-z^2 + 2z^3 + 3z^4 + \dots}{z + z^2 + z^3 + z^4 + \dots} (= \frac{1}{1-z} - 1 = \frac{z}{1-z})$$

$$A - zA = (1-z)A = \frac{z}{1-z} \therefore A = \frac{z}{(1-z)^2}$$

$$\frac{1}{z}(F^*(z) - f_0) - F^*(z) = 2 \frac{z}{(1-z)^2} + \frac{1}{1-z}$$

$$F^*(z) - zF^*(z) = \frac{2z^2 + z(1-z)}{(1-z)^2}; F^*(z) = \frac{z(1+z)}{(1-z)^3} \therefore f_k = k^2$$

$$(Check!) (k+1)^2 - k^2 = 2k + 1$$

# Laplace Transform (s-Transform)

- For \_\_\_\_\_ functions

- $L(f) = F^*(s) \equiv \int_{-\infty}^{\infty} f(t) e^{-st} dt$

For probability density,  $f(t) = 0$  ( $t < 0$ )

Thus  $F^*(s) \equiv \int_0^{\infty} f(t) e^{-st} dt$

- (Em 3.7) Laplace transform of the PDF of exponential distribution

$$\begin{aligned} F^*(s) &= \int_0^{\infty} \lambda e^{-\lambda t} e^{-st} dt \\ &= \lambda \int_0^{\infty} e^{-(\lambda+s)t} dt \\ &= \lambda \left( \frac{1}{-(\lambda+s)} e^{-(\lambda+s)t} \Big|_0^{\infty} \right) \\ &= \lambda \left( \frac{1}{-(\lambda+s)} \times (-1) \right) = \frac{\lambda}{\lambda+s} \end{aligned}$$

# Laplace Transform (s-Transform)

□ (Em 3.8) Laplace transform of  $f(t) = 8te^{-\lambda t}$

$$\begin{aligned} F^*(s) &= \int_0^\infty 8te^{-\lambda t} e^{-st} dt \\ &= 8 \int_0^\infty te^{-(\lambda+s)t} dt \quad (* \int f(t)g'(t)dt = (f(t)g(t))|_0^\infty - \int f'(t)g(t)dt *) \end{aligned}$$

Let  $f(t) = t$  and  $g'(t) = e^{-(\lambda+s)t}$ , then

$$\begin{aligned} &= 8 \left( t \frac{e^{-(\lambda+s)t}}{-(\lambda+s)} \Big|_0^\infty - \int_0^\infty \frac{e^{-(\lambda+s)t}}{-(\lambda+s)} dt \right) \\ &= 8(0 - 0 - \left( \frac{1}{-(\lambda+s)} \frac{e^{-(\lambda+s)t}}{-(\lambda+s)} \Big|_0^\infty \right)) \\ &= 8 \left( -\frac{1}{(\lambda+s)^2} (-1) \right) = \frac{8}{(\lambda+s)^2} \end{aligned}$$

# Laplace Transform Table

## Laplace Transforms

Function	$\Leftrightarrow$	Transform
$\delta(t)$	$\Leftrightarrow$	1
$u(t)$	$\Leftrightarrow$	$\frac{1}{s}$
$\frac{t^{n-1}}{(n-1)!}$	$\Leftrightarrow$	$\frac{1}{s^n}$
$Ae^{-\alpha t}u(t)$	$\Leftrightarrow$	$\frac{A}{s + \alpha}$
$\frac{t^n}{n!} e^{-\alpha t}u(t)$	$\Leftrightarrow$	$\frac{1}{(s + \alpha)^{n+1}}$
$t^n f(t)$	$\Leftrightarrow$	$(-1)^n \frac{d^n}{ds^n} F^*(s)$
$f(t - a) \quad a \geq 0$	$\Leftrightarrow$	$e^{-as} F^*(s)$
$e^{-\alpha t}f(t)$	$\Leftrightarrow$	$F^*(s + \alpha)$
$\frac{d}{dt} f(t)$	$\Leftrightarrow$	$sF^*(s) - f(0^-)$
$\int_{-\infty}^t f(t) dt$	$\Leftrightarrow$	$\frac{F^*(s)}{s} + \frac{1}{s} \frac{d}{dt} f(t) \Big _{t \rightarrow 0^-}$

Note:  $\delta(t)$  is the unit impulse function centered at 0 and  $u(t)$  is the unit step function at 0.

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# Inverse Laplace Transform

- Table look-up technique

- (Em 3.9) Inverse of  $F^*(s) = \frac{(\lambda + \mu)s + 2\lambda\mu}{s^2 + \lambda s + \mu s + \lambda\mu}$

$$F^*(s) = \frac{(\lambda + \mu)s + 2\lambda\mu}{(s + \lambda)(s + \mu)}$$

$$= \frac{A}{s + \lambda} + \frac{B}{s + \mu}$$

$$(A = \underline{\hspace{2cm}} B = \underline{\hspace{2cm}})$$

$$= \frac{\underline{\hspace{2cm}}}{s + \lambda} + \frac{\underline{\hspace{2cm}}}{s + \mu}$$

$$f(t) = \underline{\hspace{2cm}} e^{-\lambda t} + \underline{\hspace{2cm}} e^{-\mu t}$$

# Differential Equation

□  $\frac{d}{dt} f(t) \Leftrightarrow sF^*(s) - \lim_{t \rightarrow 0} f(t)$

$$\int_0^t f(\tau) d\tau \Leftrightarrow \frac{1}{s} F^*(s)$$

□ (ex)  $\frac{df(t)}{dt} + 3 \int_0^t f(\tau) d\tau = -4f(t)$

$$sF^*(s) - f(0) + \frac{3}{s} F^*(s) = -4F^*(s)$$

$$F^*(s) \left(s + \frac{3}{s} + 4\right) = f(0)$$

$$F^*(s) = f(0) \frac{s}{s^2 + 4s + 3} = f(0) \frac{s}{(s+1)(s+3)} = f(0) \left(\frac{A}{s+1} + \frac{B}{s+3}\right)$$

$$A = \underline{\hspace{2cm}} \quad B = \underline{\hspace{2cm}}$$

$$= f(0) \left(\frac{1}{s+1} + \frac{3}{s+3}\right)$$

$$= f(0) (\underline{\hspace{2cm}} e^{-t} + \underline{\hspace{2cm}} e^{-3t})$$

# Generating Functions

## □ Special properties of transforms

$$F^*(z)|_{z=0} = \sum_{k=0}^{\infty} f_k z^k|_{z=0} = f_0$$

$$F^*(z)|_{z=1} = \sum_{k=-\infty}^{\infty} f_k z^k|_{z=1} = \sum_{k=-\infty}^{\infty} f_k = 1$$

$$F^*(s)|_{s=0} = \int_0^{\infty} f(t) e^{-st} dt|_{s=0} = \int_0^{\infty} f(t) dt = 1$$

## □ Expectations

$$(-1)^n \frac{d^n F^*(s)}{ds^n} \Leftrightarrow t^n f(t)$$

$$z \frac{dF^*(z)}{dz} \Leftrightarrow kf_k$$

## Generating Functions(Cont'd)

□  $F^*(z) = f_0 + f_1 z + f_2 z^2 + \dots$

$$\frac{dF^*(z)}{dz} = f_1 + 2f_2 z + 3f_3 z^2 + \dots$$

$$\left. \frac{dF^*(z)}{dz} \right|_{z=1} = f_1 + 2f_2 + 3f_3 + \dots = \sum_{k=0}^{\infty} k f_k = E[K]$$

$$\frac{d^2 F^*(z)}{dz^2} = 2 \times 1 f_2 + 3 \times 2 f_3 z + 4 \times 3 f_4 z^2 + \dots$$

$$\left. \frac{d^2 F^*(z)}{dz^2} \right|_{z=1} = 2 \times 1 f_2 + 3 \times 2 f_3 + 4 \times 3 f_4 + \dots$$

$$\left. \frac{d^2 F^*(z)}{dz^2} \right|_{z=1} + E[K]$$

$$= 2 \times 1 f_2 + 3 \times 2 f_3 + 4 \times 3 f_4 + \dots + f_1 + 2f_2 + 3f_3 + \dots$$

$$= f_1 + 2(1+1)f_2 + 3(2+1)f_3 + 4(3+1)f_4 + \dots$$

$$= \sum_{k=1}^{\infty} k^2 f_k$$

$$= E[K^2]$$

## Generating Functions(Cont'd)

□  $E[T^n] = (-1)^n \frac{d^n F^*(s)}{ds^n} \Big|_{s=0}$

□ (Em 3.12)  $f_k = (1-p)^{k-1} p$

$$\begin{aligned}F^*(z) &= \frac{pz}{1-(1-p)z} \\E[K] &= \frac{d}{dz} F^*(z) \Big|_{z=1} \\&= \frac{p(1-(1-p)z) + pz(1-p)}{(1-(1-p)z)^2} \Big|_{z=1} \\&= \frac{p^2 + p - p^2}{p^2} = \frac{1}{p}\end{aligned}$$

## Generating Functions(Cont'd)

□ (Em 3.11) Find the average of r.v.  $T$  of exponential distribution

$$f(t) = \lambda e^{-\lambda t} \quad F^*(s) = \frac{\lambda}{\lambda + s}$$

$$\begin{aligned} E[T] &= -\frac{d}{ds} F^*(s) \Big|_{s=0} \\ &= -\lambda \left( \frac{-1}{(\lambda + s)^2} \Big|_{s=0} \right) = \frac{1}{\lambda} \end{aligned}$$

□ (Ex 3.6) Find the average of r.v.  $K$  of Poisson distribution

$$F^*(z) = e^{\lambda T(z-1)}$$

$$\begin{aligned} E[K] &= \frac{d}{dz} F^*(z) \Big|_{z=1} \\ &= \lambda T e^{\lambda T(z-1)} \Big|_{z=1} = \lambda T \end{aligned}$$

$$E[K^2] = \frac{d^2}{dz^2} F^*(z) \Big|_{z=1} + E[K] = \underline{\hspace{2cm}} + \lambda T$$

# Sum of Independent Random Variables

- $U \equiv V + W$ ,  $V$  and  $W$  are independent

$$\left. \begin{array}{l} \frac{d}{du} P[U \leq u] = f(u) \\ \frac{d}{dv} P[V \leq v] = g(v) \\ \frac{d}{dw} P[W \leq w] = h(w) \end{array} \right\} \begin{aligned} f(u) &= \int_{-\infty}^{\infty} g(v)h(u-v)dv \\ &= \int_{-\infty}^{\infty} g(u-w)h(w)dw \end{aligned}$$

$$\left. \begin{array}{l} P[U = u] = f_u \\ P[V = v] = g_v \\ P[W = w] = h_w \end{array} \right\} \begin{aligned} f_k &= \sum_{n=-\infty}^{\infty} g_n h_{k-n} \\ &= \sum_{n=-\infty}^{\infty} g_{k-n} h_n \end{aligned} \quad (\text{Convolution})$$

- Convolution Theorem

$$f(t) \otimes g(t) \Leftrightarrow F^*(s) G^*(s)$$

$$f_k \otimes g_k \Leftrightarrow F^*(z) G^*(z)$$

# Sum of Independent Random Variables(Cont'd)

□ (ex)  $h_k = f_k \otimes g_k, f_k = g_k = \begin{cases} 1 & 0 \leq k \leq 2 \\ 0 & \text{Otherwise} \end{cases}$

$$h_k = \sum_{n=-\infty}^{\infty} f_n g_{k-n}$$

	$n$	-3	-2	-1	0	1	2	3	4	5	$h_k$
$k$	루	0	0	0	1	1	1	0	0	0	$0(k < 0)$
0	$g(0-n)$	0	1	1	1	0	0	0	0	0	
	$f_n g(0-n)$	0	0	0	1	0	0	0	0	0	1
1	$g(1-n)$	0	0	1	1	1	0	0	0	0	
	$f_n g(1-n)$	0	0	0	1	1	0	0	0	0	2
2	$g(2-n)$	0	0	0	1	1	1	0	0	0	
	$f_n g(2-n)$	0	0	0	1	1	1	0	0	0	3
3	$g(3-n)$	0	0	0	0	1	1	1	0	0	
	$f_n g(3-n)$	0	0	0	0	1	1	0	0	0	2
4	$g(4-n)$	0	0	0	0	0	1	1	1	0	
	$f_n g(4-n)$	0	0	0	0	0	1	0	0	0	1
5	$g(5-n)$	0	0	0	0	0	0	1	1	1	
	$f_n g(5-n)$	0	0	0	0	0	0	0	0	0	$0(k \geq 5)$

## Sum of Independent Random Variables(Cont'd)

- If  $h_k, f_k$ , and  $g_k$  are PDF's, then

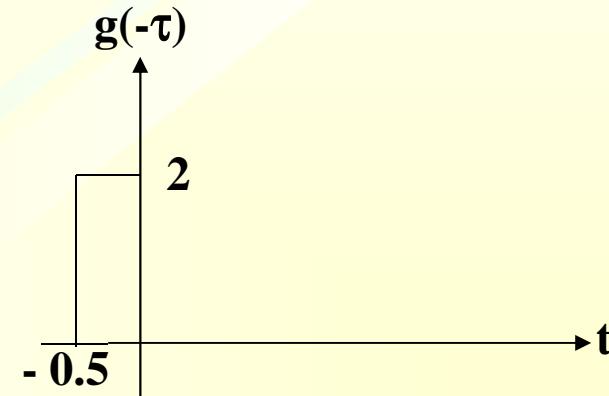
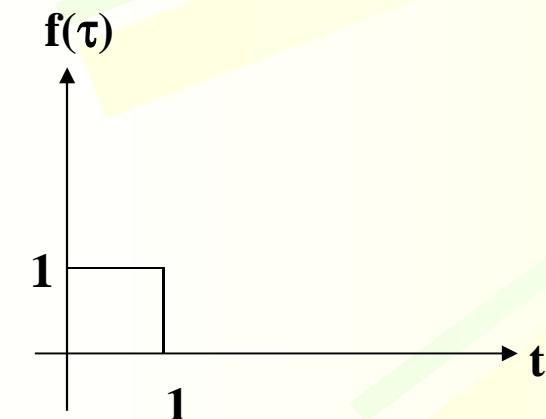
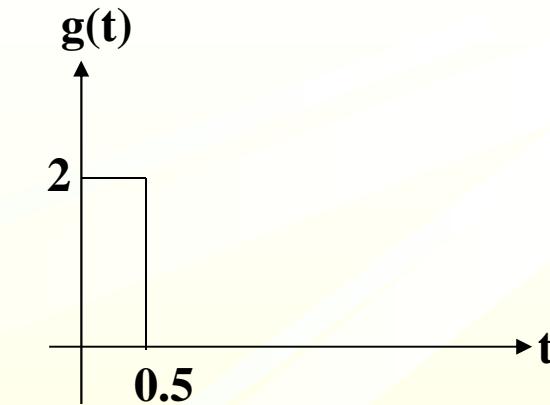
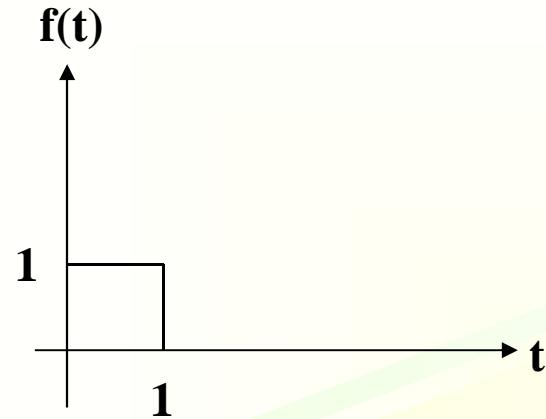
$$f_k = g_k = \begin{cases} \frac{1}{3} & 0 \leq k \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{k=0}^{\infty} f_k = \sum_{k=0}^{\infty} g_k = 1$$

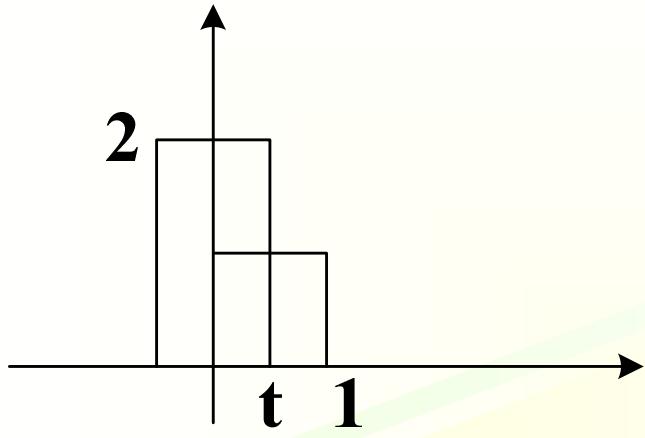
$$h_0 = \frac{1}{9}, h_1 = \frac{2}{9}, h_2 = \frac{3}{9}, h_3 = \frac{2}{9}, h_4 = \frac{1}{9}$$

$$\sum_{k=0}^{\infty} h_k = 1$$

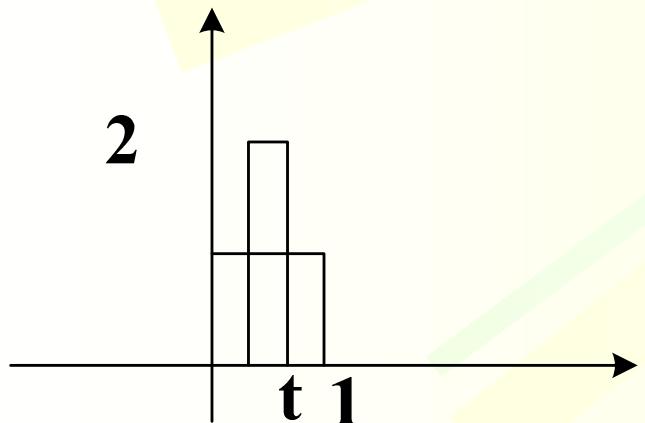
# Example of Convolution



## Example of Convolution(Cont'd)

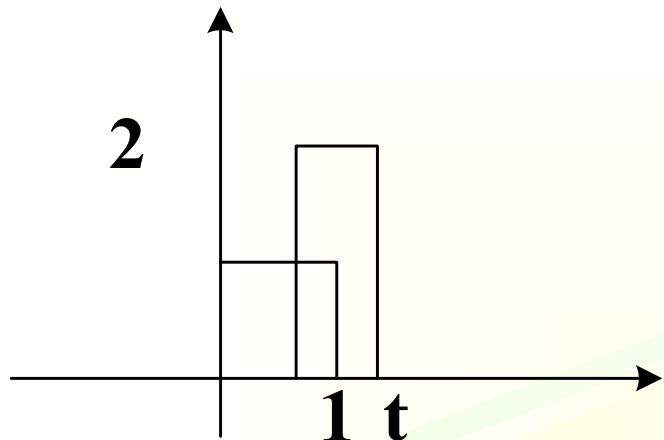


$$\int_0^t f(\tau)h(t-\tau)d\tau \\ = \int_0^t 1 \times 2 d\tau = 2t, \quad 0 \leq t \leq 0.5$$

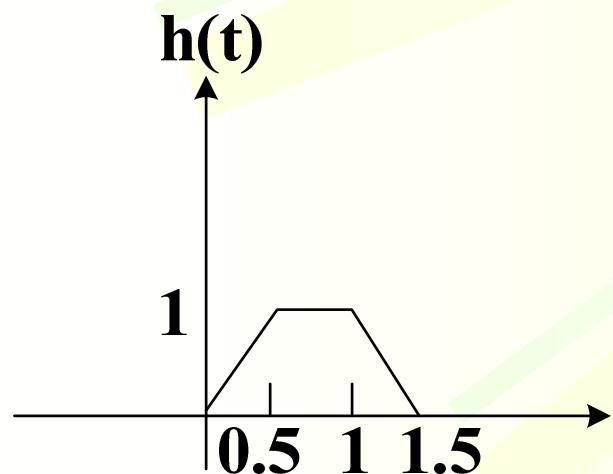


$$\int_{t-0.5}^t f(\tau)h(t-\tau)d\tau \\ = \int_{t-0.5}^t 1 \times 2 d\tau = 1, \quad 0.5 \leq t \leq 1$$

## Example of Convolution(Cont'd)

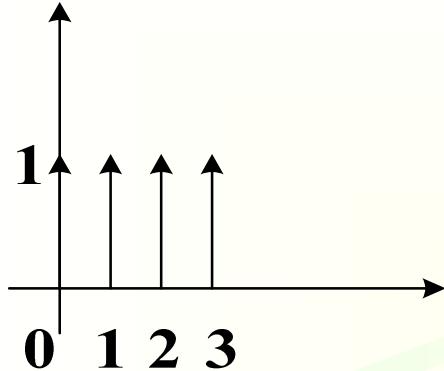


$$\begin{aligned} & \int_{t-0.5}^1 f(\tau)h(t-\tau)d\tau \\ &= \int_{t-0.5}^1 1 \times 2 d\tau = 2(1.5-t), \quad 1 \leq t \leq 1.5 \end{aligned}$$

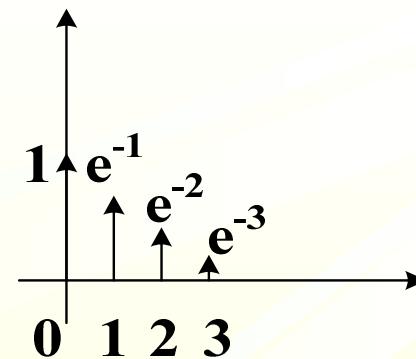


## Example of Convolution(Cont'd)

$$f_1(k) = u(k) - u(k-4)$$

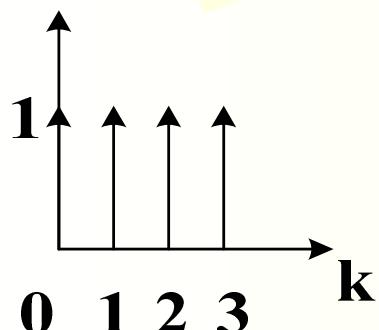


$$f_2(k) = e^{-k} u(k)$$



$$f_1(n)$$

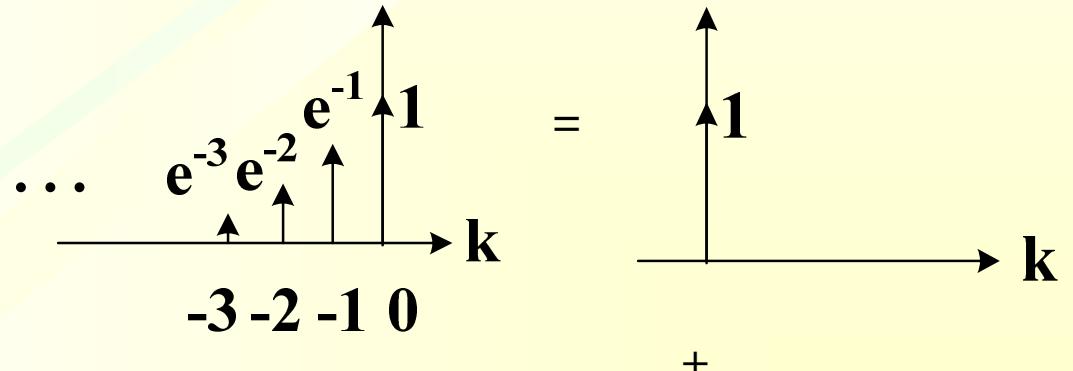
for  $k = 1$



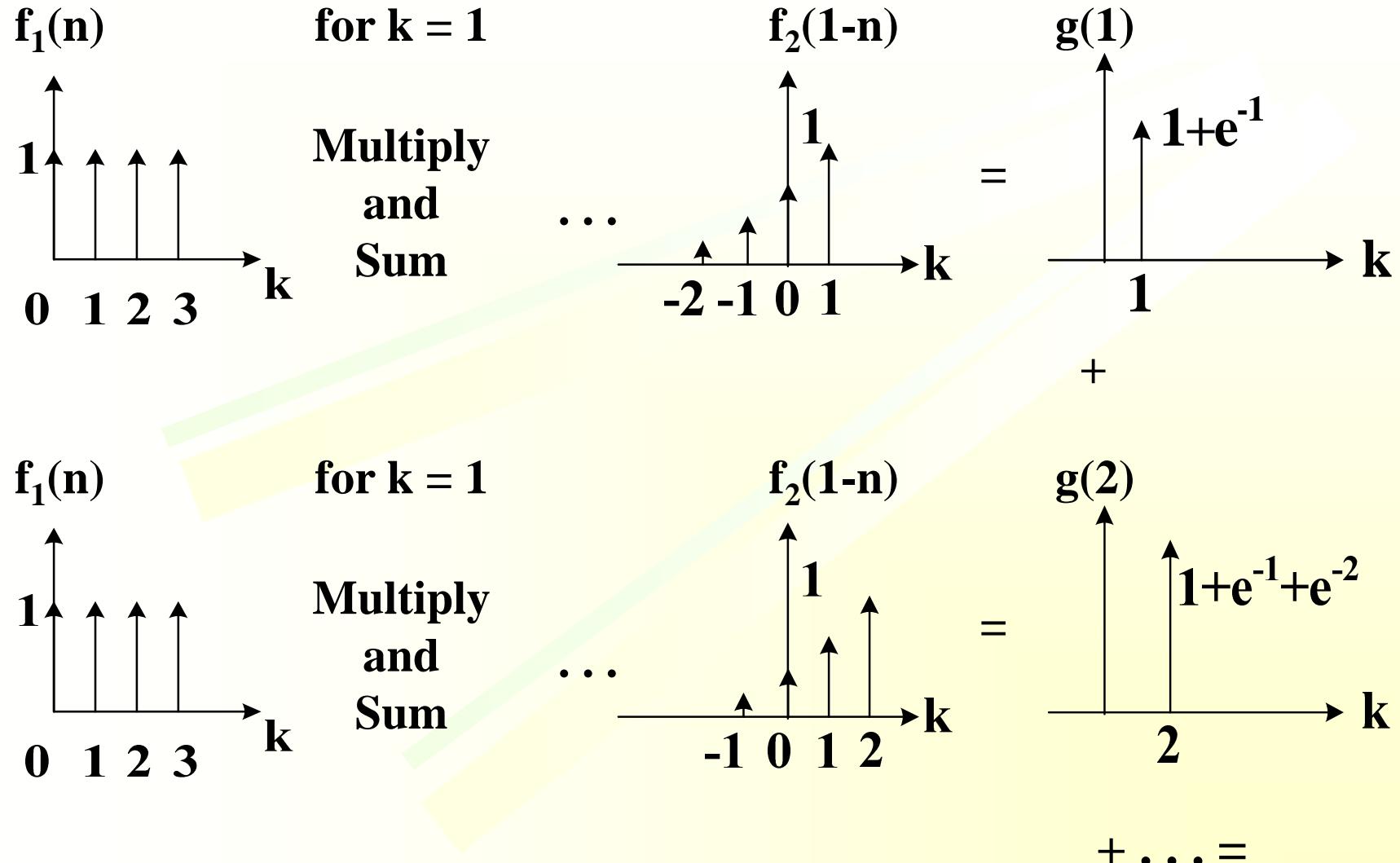
Multiply  
and  
Sum

$$f_2(-n)$$

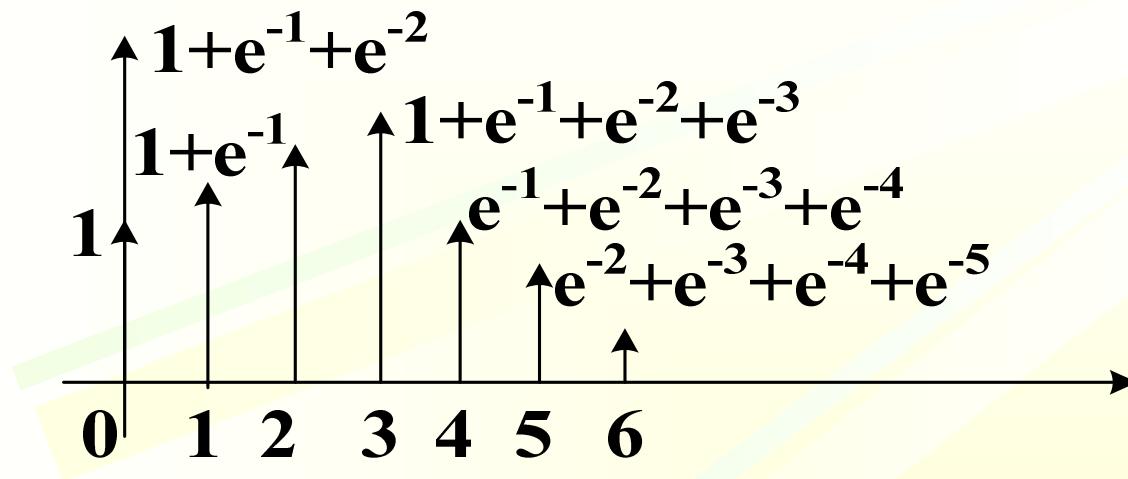
$$g(0)$$



## Example of Convolution(Cont'd)

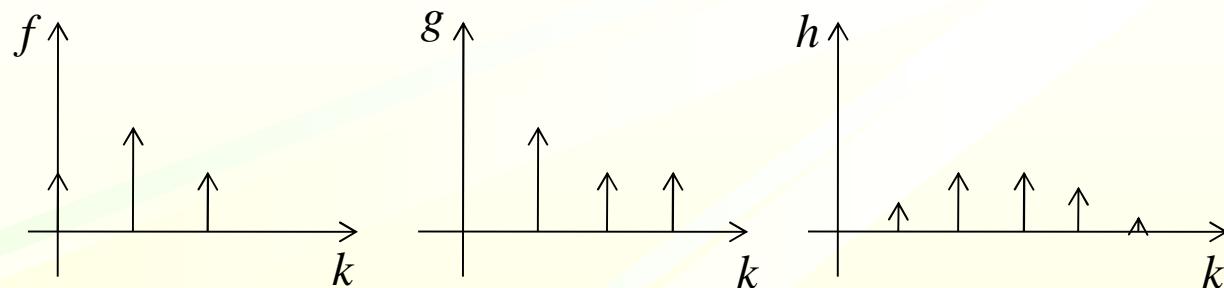
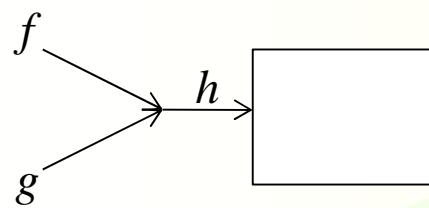


## Example of Convolution(Cont'd)



# Meaning of Convolution

- Two independent sources of input,  $f$  and  $g$ , are applied to a component, where  $f_0 = 1/4, f_1 = 1/2, f_2 = 1/4, g_1 = 1/2, g_2 = 1/4, g_3 = 1/4$ . What is the distribution of the combined input,  $h$ , to the component?



- $h_0 = P[H=0] = P[F=0]P[G=0] + P[F=1]P[G=-1] + \dots = 0$
- $h_1 = P[H=1] = P[F=0]P[G=1] + P[F=1]P[G=0] + P[F=2]P[G=-1] \dots = 1/8$
- $h_2 = P[H=2] = P[F=0]P[G=2] + P[F=1]P[G=1] + P[F=2]P[G=0] \dots = 5/16$
- $h_3 = P[H=3] = P[F=0]P[G=3] + P[F=1]P[G=2] + P[F=2]P[G=1] \dots = 5/16$
- $h_4 = P[H=4] = P[F=0]P[G=4] + P[F=1]P[G=3] + P[F=2]P[G=2] \dots = 3/16$
- $h_5 = P[H=5] = P[F=1]P[G=4] + P[F=2]P[G=3] + P[F=3]P[G=2] \dots = 1/16$
- $h_6 = P[H=6] = P[F=2]P[G=4] + P[F=3]P[G=3] + \dots = 0$

# Superposition

- Two independent sources are Poisson distribution,  $f$  and  $g$ , with the rate of  $\lambda_1 T$  and  $\lambda_2 T$ , respectively. What is the distribution of the combined input,  $h$ ?

$$F^*(z) = e^{-\lambda_1 T(1-z)} \quad G^*(z) = e^{-\lambda_2 T(1-z)}$$

By convolution

$$H^* = F^*(z)G^*(z)$$

$$= e^{-(\lambda_1 + \lambda_2)T(1-z)}$$

$$h_k = \frac{(\lambda_1 + \lambda_2)T}{k!} e^{-(\lambda_1 + \lambda_2)T} \quad (* \text{ Superposition } *)$$

# Unconditioning

□ To uncondition a transform without inverting (get  $F^*(s)$  from  $F^*(s | r)$ )

$$\square f(t) = \int_0^\infty f(t/r)g(r)dr$$

$$\begin{aligned}\square F^*(s) &= \int_0^\infty [\int_0^\infty f(t/r)g(r)dr] e^{-st} dt \\ &= \int_0^\infty [\int_0^\infty f(t/r)e^{-st} dt] g(r)dr \\ &= \int_0^\infty [F^*(s/r)] g(r)dr\end{aligned}$$

$$\square f_k = \sum_{n=0}^{\infty} f_{k/n} g_n$$

$$\begin{aligned}\square F^*(z) &= \sum_{n=0}^{\infty} f_k z^k \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f_{k/n} g_n z^k \\ &= \sum_{n=0}^{\infty} [\sum_{k=0}^{\infty} f_{k/n} z^k] g_n \\ &= \sum_{n=0}^{\infty} [F^*(z/n)] g_n\end{aligned}$$